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Teachers' Collective Noticing of Children's Mathematical Thinking in Self-facilitated Collaborative Inquiry

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Teachers’ Collective Noticing of Children’s Mathematical Thinking in Self-facilitated Collaborative Inquiry

by

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Dedication

In memory of Tom Carpenter. Many thanks to you and your esteemed colleagues, including Elizabeth Fennema, for establishing a space for an amazing group of educators. I am so fortunate to have been welcomed in to this community and in awe of the work you have inspired over the decades.

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Now, who wants some home baked cookies? Or perhaps a cake, or pie?

Abstract

Teachers' Collective Noticing of Children's Mathematical Thinking in Self-Facilitated Collaborative Inquiry

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The University of Texas at Austin, 2018

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Many educators assume the value of teachers working together and collaboration has the potential to help teachers learn from each other and develop their practice (Crockett, 2002). Although the general value for teachers of participating in collaborative inquiry groups has been established, working together does not guarantee that opportunities for learning and development will be created (Horn & Kane, 2015). The purpose of my qualitative research study is to examine the kinds of opportunities teachers create to notice children's mathematical thinking as they interact with one another in self-facilitated collaborative inquiry groups intended to support teachers in their development of professional noticing. Research suggests that the practice of noticing children's mathematical thinking is a learnable, but complex skill that takes time — often years — to develop (Jacobs et al., 2010). Using techniques from discourse analysis, my findings suggest that teachers participating in self-facilitated collective inquiry not only have the potential to support one another in noticing, but can also take an opportunity to jointly construct a student strategy, perhaps helping teachers to engage in more complete descriptions of student thinking. When teachers participate

in discussions that are grounded in the details of student strategies, they have an opportunity to continue to develop expertise in their noticing of children's mathematical thinking through the articulation and reflection of children's mathematical thinking.

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Chapter 1 : Introduction

Researcher: We are interested in what you learn from analyzing students' written work.

Teacher 1: I think it showed me, and I don't know how feasible it would be, because taking home student work and laying it out and analyzing it every night is time-consuming; it's worth it, but it's time-consuming, but it helped to sit down with a group and just—because they're going to—different people are going to see different things. I feel that's mainly, probably, one of my biggest faults, is I sit down, like, "Oh, that person did that, that person did that." Because it is a time-consuming thing, so you don't really delve into it as much as you should.

Teacher 2: Right, like you look at something and, like, "OK, they did this." But maybe if you looked at it a little bit longer, you could see—.

Teacher 1: Or somebody else might see a little something else, you know—.

Teacher 3: Yeah, I think, when I just look and glance at work, it's—I just assume one thing. But then having that conversation around it with other people, I notice different things once people start bringing up what they notice. It's more helpful to do it in a group than independently.

— Focus group conversation after one week of professional development

Many educators assume the value of teachers working together and indeed, as the teacher in the quote above explained, when teachers get together with other teachers to examine students' written work for mathematics problems, they may "notice different things once people start bringing up what they notice." Teachers' collaborative work has the potential to help teachers learn from each other and develop their practice (Crockett, 2002) and many studies have documented this potential (Amador & Carter, 2018; Levine & Marcus, 2010; Kazemi & Franke, 2004, Sherin & Han, 2004; van Es & Sherin, 2006; Westheimer, 2008).

Although the general value for teachers of participating in collaborative inquiry groups have been established, working together does not guarantee that opportunities for learning and development will be created (Horn & Kane, 2015). Whether teachers learn and develop as a result of their collaborative inquiry can depend on a number of factors, including in particular teachers' conversational routines and patterns of interactions in these groups. Researchers have recently begun to document and explain how teachers' interactions with each other in these work groups can open up opportunities to develop practice and learn from each other (Bannister, 2018; Crespo, 2006; Horn et al., 2017; Little, 2003). Examples of these interactions include: (a) clarifying details, (b) offering alternative perspectives, (c) refining and negotiating what the details of a student strategy reveal about student understanding, and (d) discussing instructional implications based on those understandings.

However, interactions that open up opportunities for teachers in particular can be rare, especially in self-facilitated conversations (Horn et al., 2017), and therefore it is important to understand how and under what circumstances teachers' self-facilitated work in collaborative groups might be productive — that is, offer opportunities for teachers to learn and develop professionally in their work with colleagues. Understanding how teachers' work together can be productive will inform the design of future collaborative inquiry groups and the supports that teachers might need to engage with one another.

For this study I examined teachers' self-facilitated collaborative-inquiry in which elementary-grade teachers met together in school-based groups to work on developing expertise in noticing children's mathematical thinking. Professional noticing of children's mathematical thinking refers to the practice of, almost simultaneously, making sense of children's mathematical thinking during instruction and deciding how to respond to that thinking (Jacobs, Lamb, & Philipp, 2010). Therefore, this practice is foundational to teaching in ways that are responsive to children's mathematical thinking, or instruction that builds on children's understanding. Research suggests that the practice of noticing children's mathematical thinking is a learnable, but complex skill that takes time — often years — to develop (Jacobs et al., 2010). The development of this practice is therefore an important goal in professional development centered on supporting teachers to learn to teach in ways that are responsive to children's mathematical thinking.

Professional noticing is a generative teaching practice in that teachers have opportunities to continually refine both what and how they notice the mathematical thinking of their students on the basis of their own noticing, inside and outside of the classroom. When teachers meet together outside of classroom teaching to discuss students' written work, they have an opportunity to reflect on their noticing by making explicit what they notice, based on the written work, about a student's mathematical thinking not only to their partner teacher, but also to themselves. I refer to these discussions as collective noticing, to distinguish them from the noticing that individual teachers engage in during instruction. In addition, when multiple teachers discuss the same piece of student work, each teacher has an opportunity to consider and voice her or his perspective or potential interpretation of the student work, which could allow for robust professional noticing, or discussions that consider a range of possibilities that are grounded in the details of a student strategy. Participation in collective noticing affords teachers an opportunity to develop their own practice of noticing.

The purpose of my study is to examine the kinds of opportunities teachers create to notice children's mathematical thinking as they interact with one another in self-facilitated collaborative inquiry groups intended to support teachers in their development of professional noticing. The collaborative inquiry sessions were considered self-facilitated in that teachers used a protocol to facilitate their own discussions of student work outside of professional development.

Methods and Research Question

In this qualitative research study (Miles, Huberman, & Saldaña, 2014), I explored how 3rd–5th grade teachers worked together in self-facilitated collaborative inquiry groups designed to provide opportunities to develop their capacity to notice children's mathematical thinking (Jacobs et al., 2010). In particular, I examined not only what teachers collectively noticed about the mathematical thinking of their students, but also how teachers' interactions, as they worked together, opened up or constrained opportunities to notice their students' mathematical thinking. The data for this study were taken from a larger professional-development design study, the Responsive Teaching in Elementary Mathematics (RTEM) Project, in which the overall goal was to study the characteristics of responsive teaching, how to support its development in the domain of fractions, and how it is related to student learning gains (Empson & Jacobs, 2012). The teachers in this study participated in a three-year professional development (PD) program designed to support teachers in understanding how children develop understanding of fractions and in developing teaching practices that are responsive to children's progressive understanding. Teachers were required to meet face-to-face four times per year in groups comprising two to four teachers, preferably from their own campus, with the goal of continuing discussions about what their students understood and what their next instructional steps could be, or developing collective noticing.

To support teachers' participation in self-facilitated collaborative inquiry in noticing children's mathematical thinking, the RTEM project team designed a web-based tool consisting of 13 online collaborative-inquiry sessions based on research on how children think about and solve problems (Carpenter, Fennema, Franke, Levi, & Empson, 2014; Empson & Levi, 2011). The sessions were designed to help teachers connect what they were learning in the PD setting, which may be perceived as decontextualized from the classroom, to their own practice. Before meeting, teachers posed a common problem to their students suggested by the tool, collected the written work from the classroom, and brought at least three pieces of student work to discuss with their colleagues.

I analyzed audio recordings of teachers' collaborative inquiry sessions using discourse analysis techniques, specifically borrowing from conversation analysis (Schegloff, 2007), to explore how teachers' interactions in self-facilitated groups allowed for engagement in collective noticing of children's mathematical thinking and how these interactions opened up the groups' opportunities (Little, 2003) to notice children's mathematical thinking. This study contributes to still-needed research into the features of teacher collaboration that may enhance teachers' development and its findings have implications for how to create support structures that facilitate teachers in this process (Hindin, Morocco, Mott, & Aguilar, 2007; Kennedy, 2016; Slavit, Kennedy, Lean, Nelson, & Deuel, 2011; Vangrieken, Dochy, Raes, & Kyndt, 2015).

Outline of the Dissertation

In Chapter Two, I outline my conceptual framework with a review of the literature on collaborative inquiry groups, patterns of conversational interactions within these groups and teacher noticing in the domain of rational numbers. In Chapter Three, I provide a description of the Collaborative Inquiry Tool, using the fifth module as an example. In Chapter Four, I describe my study design and use of discourse analysis to analyze the audio recordings of teacher sessions. In Chapter Five, I present the major findings and results of the study. Lastly, in Chapter Six, I conclude with a discussion of my findings, current methodological limitations, and future directions of this work.

Chapter 2 : Conceptual Framework

Within professional development, and in particular the Collaborative Inquiry sessions, instructional practices that are core to responsive teaching, such as the professional noticing of children's mathematical thinking and asking questioning to support and extend children's mathematical thinking (Jacobs & Empson, 2015), are practiced outside of the teachers' classrooms. Engaging in these practices outside of the classroom context, where the demands for a teacher's attention can be put aside, provided teachers with an opportunity to slow down and reflect on instructional decisions and the evidence they considered in these decisions. My study explored how teachers collaborated with one another to engage in the practice of noticing. In this chapter, I discuss the conceptual framework and present a review of relevant research that underlies the analysis of my study.

To develop this framework, I reviewed research on teachers working together for the purposes of professional development in, mostly facilitated, collaborative inquiry groups and the interactional patterns that are associated with productive sessions, as well as the research on the development of teachers' capacity to notice children's mathematical thinking. The intersection of collaboration and professional noticing informed my analysis of how teachers worked together to engage in the practice of professional noticing by examining the substance of what teachers noticed about the mathematical thinking of their students and the patterns of interactions that teachers used in collaborative inquiry.

I begin with literature on collaborative inquiry as a context for professional development. I then examine theoretical and empirical research on collaboration and the contribution of ideas.

Next, I situate the focus of teacher collaboration in my study by presenting research on the professional noticing of children's mathematical thinking.

Collaborative Inquiry

School-based collaborative inquiry groups are increasingly recommended as a key feature of teacher professional development and allow teachers to learn from one another through discussions of teaching and learning. Ideally, they "support teachers in making decisions based on their contexts, their goals, current and new professional knowledge, and the needs of their students" (Vescio, 2008, p. 89), and allow teachers to reflect on these decisions publicly with their peers, a practice typically done in isolation (Little, Gearhart, Curry, & Kafka, 2003). Research has shown that in collaborative inquiry groups teachers have opportunities to develop their practice (Bannister, 2015; Butler & Schnellert, 2012; Cochran-Smith & Lytle, 1999; Crockett, 2002; Horn, 2005; Horn & Little, 2010; Nelson & Slavit, 2008); however, Levine and Marcus (2010) argue that these opportunities are affected by the structure and focus of the collaboration. The structure of a collaborative inquiry can include the presence of a facilitator or participant roles, a structured protocol to outline the discussion, the frequency of the meetings, and a specific purpose. Within the collaboration, teachers can focus on curriculum and instruction, analyzing individual or group-level student data, or school management–related discussions, such as policies and routines. For the purposes of my study, teachers met in self-facilitated sessions, guided by a protocol, to discuss their students' mathematical thinking.

Structures of Collaborative Inquiry

In addition to setting and communicating clear purposes for engaging in collaborative inquiry, it is important to consider how the structures of collaborative-inquiry groups can potentially open or constrain teachers' opportunities to develop in their practice (Little, 2003). Collaborations can include a facilitator and/or a structured protocol to help teachers attend to the task.

Presence of a facilitator. Research on collaborative inquiry groups has also attended to whether a group was facilitated by someone with more knowledge and the nature of this facilitation. I have identified two main types of facilitation: facilitation by a more knowledgeable other and self-facilitation, in which peer teachers regulate their conversation related to the task. When teacher collaborative inquiry groups are investigating their own practice, often an outsider such as a facilitator is present in order to maintain the focus of the conversation and provide additional perspectives.

However, depending on additional job requirements, facilitators may not always be available to attend scheduled meetings. For example, Slavit and Nelson (2010) reported in a case study that an assigned facilitator was only able to attend one meeting every six weeks, which was fewer meetings than had been planned. The researchers and the facilitator posited that this may have contributed to the group's difficulty in focusing on student thinking throughout the year even with a facilitator present. The researchers found that teachers often made generalizations about student work without providing specific evidence, even when the facilitator enacted moves to push teachers to elaborate on these details. In addition, researchers have examined if and how teachers begin to take ownership in productive discussion practices (van Es et al., 2014) while a facilitator is present. When Horn and Kane (2015) studied the discussion patterns of teacher groups, they found more variation in the types of discussion patterns among the groups that were self-facilitated, suggesting that some teacher groups are better able to engage in sustained and productive discussions, implementing support structures could lead to more productive and in-depth discussions for some groups. This finding in particular is important to note when considering how to promote focused and sustained discussions of students' mathematical thinking in a way that provides teachers with the opportunity to continually develop their professional noticing of children's mathematical thinking.

Other research on self-facilitated groups found that teachers tended not to engage in critical reflection (Louie, 2015; Vangrieken et al., 2015), as teachers do not necessarily take the opportunity to discuss multiple perspectives (Hindin et al., 2007), possibly in an effort to avoid conflict (Achinstein, 2002; Levine & Marcus, 2010).

Furthermore, Hindin et al. (2007) found that when teachers met to plan for and review the effectiveness of curricular units, teachers tended to spend more time talking about potential tasks for upcoming lessons, rather than using previous student work to examine student thinking to inform instructional decisions.

However, while research suggests the importance of facilitators in leading collaborative inquiry, Bannister (2015) found that teachers' participation patterns can change when working in self-facilitated collaborative-inquiry groups over an academic year, and Horn and Kane (2015) posited that self-facilitated groups can engage in productive and sophisticated discussions about teaching practices when teachers have greater knowledge regarding the topic being discussed. These findings suggest that with the appropriate structures in place, self-facilitated collaborative inquires can be a productive and generative activity for teachers.

Still, these collaborative-inquiry groups vary in effectiveness and further studies should be conducted to examine what teachers discuss in these settings (Kennedy, 2016; Levine & Marcus, 2010). In addition, Franke, Carpenter, Levi, and Fennema (2001) found that while teachers believed the support of their colleagues was critical in their own development, one teacher, who had been able to discuss student thinking with a facilitator, shared "I'm really not sure it's the bouncing of what kids are doing with another colleague as much as I think it helps to bounce it off someone who really has knowledge about kids' thinking," (p. 681) indicating that the teacher believed discussions with specific people were more productive than with others. However, while a more knowledgeable other may be appreciated, when teachers engage in self-facilitated collaborative inquiry around student work, they have an opportunity to sustain conversations that are grounded in the details of the students' strategies, develop norms for offering their own perspectives, and consider questions they could pose to learn more about the student's thinking, such as asking the student questions about his or her thinking.

Structured Protocols. Protocols can be used as a tool to structure collaborative inquiry discussions in order to facilitate opportunities for teachers to explore issues of teaching and learning (Curry, 2008; Nelson & Slavit, 2008; Nelson, Slavit, Perkins, & Hathorn, 2008). Protocols can structure a discussion in phases, providing prompts and suggested lengths for each phase. When examining teacher discussions in collaborative inquiry sessions, Levine and Marcus (2010) found three key features of protocol use: (a) teachers were prompted to discuss their teaching practices with one another; (b) teachers had an opportunity to determine the content they shared; and (c) the way teachers framed their inquiry may depend on the types of prompts the different protocols contained.

Protocols also allow teachers to connect their ideas with one another (Kintz et al., 2015) and elaborate on their ideas using more specific details (Levine & Marcus, 2010). However, Little et al. (2003) found that protocols were not sufficient for encouraging productive discussions and that enacting protocols in ways that promote further reflection and discussion takes time to develop. For example, Bannister (2015) found that when teachers first began enacting a specific protocol, they often allowed each speaker to take a turn and the others rarely took the opportunity to ask the speaker follow-up questions. However, over time the enactment of the protocol changed as the teachers began to interrupt one another, asking for more elaboration. Therefore, after they become familiar with a protocol, a reasonable expectation might be that teachers begin to use protocols more as guidelines and make adjustments as needed (Curry, 2008; Little et al., 2003; Wood, 2007) to pursue conversations that encourage deeper reflections.

While many researchers have documented the benefits of structured protocols, Curry (2008) cautions that discussions can be constrained as teachers may use protocols as a checklist, moving to the next agenda item rather than sustaining conversations related to practice.

Purposes of Collaborative Inquiry

Kintz, Lane, Gotwals, and Cisterna (2015) found that collaborative inquiry groups either tended to use their time focused on a single purpose or multiple purposes. Meetings became less productive when teachers were expected to discuss three or more agenda items (Curry, 2008). When teachers participated in collaborative inquiry for a single purpose, they had an opportunity to engage in sustained discussions that promoted reflection and in-depth analysis. However, sustained conversation was a necessary but not sufficient factor that contributed to deeper discussions.

Professional Development. When teachers meet in collaborative inquiry groups they have an opportunity to investigate and reflect on their teaching practices within their own professional contexts (Cochran-Smith & Lytle, 2004). Researchers have typically conceptualized collaborative inquiry as a cycle of inquiry in which teachers determine the focus for the cycle, collect student data to discuss and analyze, and discuss implications of their findings for their practice (Ciampa & Gallagher, 2016; Crockett, 2002; Nelson & Slavit, 2008; Zech, Gause-Vega, Bray, Secules, & Goldman, 2000). However, for the purposes of my study, the focus of both a macrocycle (engagement with and completion of all sessions) and the microcycle (each individual session) were set by the researchers and communicated via the online tool.

Focus on teaching practices and student learning. When teachers meet in collaborative inquiry groups, to discuss their everyday work of teaching (Levine & Marcus, 2010; Wood, 2007) teachers have an opportunity to use data from their classrooms to examine their own teaching practices (Nelson & Slavit, 2008; Supovitz & Christman, 2003). Teachers can use observation and assessment data to discuss the understandings of their students in order to design and implement instructional practices. One way teachers can connect their teaching practice to their students' learning is through the examination of student work. Goldsmith and Seago (2011) found when teachers examined student work from their own classrooms, teachers often used prior knowledge about their classrooms and their students to inform and justify their noticing of student work, rather than examining the details of a student strategy to reconstruct the student's potential reasoning, which could lead to incomplete descriptions or assumptions not supported by the evidence from the strategy.

For this reason, when teachers bring selected samples of student work to a collaborative inquiry session, there should be an alignment between the discussion of student work and their teaching practice that allows for deep connections (Kintz et al., 2015; Nelson & Slavit, 2010). When teachers are pushed to describe details of student work (Kazemi & Franke, 2004) their discussions of instructional implications are grounded in reflections of what students understand (Little et al., 2003).

However, the quality of the collaborative group can vary if the purpose for examining student and how the teachers should contribute to the group is not clear. For example, some teachers in Curry's (2008) case study reported frustration when other teachers seemed to look at superficial characteristics of student work, rather than concentrating on the substance of the students' ideas, a central concern for the teacher sharing the work. In addition, Levine and Marcus (2010) found that when teachers were not given a focus, the teachers rarely discussed how their teaching practice and decisions could impact student outcomes and achievement.

Collaborative Inquiry for the Purposes of the Study

For the purposes of this study, collaborative inquiry is used to describe a group of teachers meeting together to inquire about student understanding through the examination of student work. These self-facilitated collaborative inquiry sessions were designed to provide teachers with a space to make their own noticing of children's mathematical thinking visible to themselves and their colleagues (Goldsmith, Doerr, & Lewis, 2014). Together, teachers examined and discussed students' thinking through the use of written student work in order to develop this instructional practice in a job-embedded setting.

Patterns of Interaction in Collaborative Inquiry

Most researchers have agreed that while collaboration may be necessary, talking is not sufficient for teacher development, but rather how teachers talk is important. Teachers need opportunities to challenge or build on one another's ideas (Dobie & Anderson, 2015; Goldsmith et al., 2014; Lord, 1994) and the sharing of multiple perspectives or interpretations of children's mathematical thinking contributes to teachers' development of deeper insights of this thinking (Chamberlin, 2005). For the purposes of my study, teachers had an opportunity to verbalize their noticing to the collaborative inquiry group when they remained grounded in the details of the students' written work. In addition, teachers had an opportunity to reflect on their noticing when the group provided feedback through agreement, questions, or alternative interpretations. These patterns of interaction contribute to the opportunities teachers created to develop their professional noticing of children's mathematical thinking.

Strawson defines uptake as how the listener understands the speaker, rather than how the listener responds to speaker (as cited in Bach & Jarnish, 1982, p. 13), while Collins defines uptake as a question posed in response to something the speaker said previously (as cited in Nystrand & Gamoran, 1997, p. 36). For my own study, uptake refers to how the listener provides evidence of understanding the speaker in conversation, or how the listener contributes to the collective noticing. However, not all responses demonstrate the same type or level of evidence of understanding. Therefore, using previous studies, I characterized teachers' patterns of interaction, or responses, as opportunities to help teachers either individually or jointly notice children's mathematical thinking. In particular, I considered how teachers responded to one another, or how discussions of students' strategies were taken up, to characterize the conversational turns (Schiffrin, 1994).

Within the discussion, teachers could respond by agreeing with one another, sharing observations or experiences, providing alternative suggestions, or asking one another to justify their thinking (Chamberlin, 2004; Kintz et al., 2015; Mercer, 1995, 2000). Interactions in which teacher had the opportunity individually describe what they noticed were helpful in making this practice visible to both themselves and one another; however, this practice was enhanced when at least two teachers contributed to the descriptions.

Individual construction. Contributions in which one teacher described the mathematical thinking of a student and a second teacher supported the descriptions were considered as individual contributions. Patterns of interaction, or contributions, could take a number of forms within these discussions, such as: (a) repetition, (b) agreement, (c) clarification, and (d) no contributions.

Agreement and Repetition. One pattern of interaction related to individual contributions is agreement. Agreement is a form of assessment that can take on different forms indicating either strong or weak agreement (Liddicoat, 2011). For example, when a partner teacher demonstrates agreement, they can intend to affirm or upgrade an idea. Another way teachers can demonstrate agreement is through anticipatory completion (Lerner, 1996). Crespo (2006) claimed in her study that when teachers would interrupt or overlap speech in order to finish one another's sentences or ideas, they were demonstrating intellectual involvement with one another.

When partner teachers repeat a phrase or utterance, they have an opportunity to check their own understanding of or demonstrate agreement with what was stated (Tannen, 1989). Mercer (2000) suggested when repetitions are used in discussions, participants (or teachers in this study) have an opportunity to create cohesion within their ideas (p. 62).

Clarification. Facilitators often ask clarifying or probing questions, which are associated with a greater depth of discussion (Kintz et al., 2015) by pressing teachers for details and reminding teachers to provide rationales for student thinking based on evidence of the strategy, rather than dismiss or make assumptions about student thinking (Andrews-Larson et al., 2017; Chamberlin, 2005). In addition, facilitators can encourage discourse among the participant teachers, asking others to add their own thoughts and experiences to the discussion (Crespo, 2006).

No contributions. If uptake is evidenced as a listener responding to a speaker in a way that demonstrates understanding, then no uptake might be considered as the absence of a response. While a lack of response, or silence, can indicate issues of power and control between the speaker and the hearer through defiance or dominance (Mercer, 2000), a lack of contribution might also refer to how a partner teacher might respond in a disjointed way.

Researchers have found that a common interaction pattern used by teachers in collaborative groups is to share their experiences without making connections to one another (Kintz et al., 2015; Louie, 2015). Kintz et al. describe these types of interaction as one-way sharing, when the topic changed after a teacher shared a contribution, and parallel sharing, when a teacher shifted the topic with little or no connection to the previous teacher's contribution. Within these interactions, the speaking teacher had an opportunity to formulate and contribute an idea, but there was no evidence to suggest how this idea was received or understood. Due to the shift of topic, these interactions are considered to be examples of no contributions, or a lack of sustained engagement with one another's ideas.

Of particular interest to my study, Crespo (2006) found when teachers reported on student work from their own classrooms, the teachers often had uninterrupted opportunities to present their analysis of the students' mathematical details, and the speaking teacher did not invite responses. Crespo went on to suggest that discussing what already happened, possibly as an authority, left little opportunity for others to share their own thinking.

Joint construction. While researchers have documented teachers' propensity to engage in congenial conversations (Achinstein, 2002; Grossman, Wineburg, & Woolworth, 2001), most agree that teacher groups have an opportunity to examine their practice through critical reflection by asking for feedback and offering differing perspectives (Hindin et al., 2007; Nelson et al., 2010).

Counterclaims and elaborations. Another pattern of interaction teachers engage in is contributing to the idea by offering their own insights or interpretations through elaborations. Elaborations can take on many forms, such as offering claims or counter claims, adding details or information, and have the potential of being supported by evidence from the student's written work, encouraging teachers to remain grounded in the details of the student's strategy.

Teacher Noticing in Mathematics Education

During instruction, there is variation in what teachers notice in the moment, such as the clothes that students are wearing, who students like to talk to, the students who are attentive, the types of questions students are asking, and the responses students provide. While what teachers notice inside of their classrooms contributes to how they make sense of their students, some observations can provide teachers with a better window into how students are understanding mathematical concepts than others. In mathematics education, professional noticing has been considered in two ways (Jacobs & Spangler, 2017): as components of teacher noticing (e.g., attention and interpretation) and types of teacher noticing (e.g., student behavior, interaction with materials or concepts, correct and incorrect answers). Recently there has been a call to support teachers in noticing issues of equity, including students' culture, identity, and how students are positioned (Henry, 2017; Louie, 2017, 2018). The practice of noticing in the moment is considered professional because it draws on specialized knowledge that teachers have about teaching and learning. It is an invisible practice that is difficult to observe because teachers are not often expected to articulate how they used what they attended to when making instructional decisions about how to respond to a student. Sherin and van Es (2009) defined noticing as (a) what events teachers identify as important to teaching in a classroom context and (b) the knowledge that teachers use to make sense of those events, while Star and Strickland (2008) restrict this definition to the classroom experiences teachers identify as important or noteworthy, or what the authors consider to be foundational to noticing. While these definitions are important to understanding the many components that influence the decisions teachers make, this study seeks to explore in particular what teachers collectively notice regarding the mathematical thinking of their students.

Children's mathematical thinking. Research has documented the importance of examining children's mathematical thinking by looking at children's strategies and engaging children in a discussion of those strategies to help teachers make sense of what their students understand (Carpenter et al., 2014; Dyer & Sherin, 2015; Jacobs et al., 2010, 2011; Jacobs & Empson, 2015; Kazemi & Franke, 2004; Sherin & van Es, 2006, 2009; Steinberg, Empson, & Carpenter., 2004). When considering how teachers notice children's mathematical thinking, I draw on the research conducted by Empson and colleagues (Empson, 1999; Empson, Junk, Dominguez, & Turner, 2005; Empson & Levi, 2011). Empson and colleagues have conducted research to characterize children's typical informal strategies for problem types that have been strategically selected to elicit how they understand rational numbers (e.g. fractions). This framework is used because teachers in the study engaged in professional development centered on developing children's understanding of fractions.

Professional noticing of children's mathematical thinking. When considering the noticing of mathematics teachers, I use the lens of Jacobs and colleagues (Jacobs et al., 2010; Jacobs, Lamb, Philipp, & Schappelle, 2011) who conceptualized the practice of professional noticing of children's mathematical thinking, which draws on research-based knowledge of children's thinking. According to their work, noticing is defined as the integration of three interrelated skills: a) attending to the details of children's strategies; b) interpreting children's understandings reflected in those strategies; and c) deciding how to respond on the basis of those understandings. As noticing children's mathematical thinking is fundamental to responsive teaching, it is important to facilitate opportunities for teachers to develop the three components in interconnected ways.

Teachers can deepen their knowledge of children's mathematical thinking and the practice of noticing children's mathematical thinking by meeting regularly and examining students' written work (Little, 2005). While teachers may initially engage in discussing what they notice about their students' mathematical thinking in vague ways and drive the conversation to the broader topic of classroom instruction, teachers can begin to engage in sustained and more focused discussions about individual student thinking over an academic year (Kazemi & Franke, 2004; van Es & Sherin, 2007). This finding suggests that engaging in discussions of children's written work could develop teachers' capacity to notice children's mathematical thinking. However, in a case study analysis with a group of secondary teachers, Slavit and Nelson (2010) found teachers continued to provide vague descriptions of student thinking and spent most of their collaborative time discussing instructional practices related to tasks even when a facilitator was present. This suggests that not all teacher groups have the same opportunities to sustain in conversation about children's mathematical thinking and the structure and focus of the collaboration could contribute to this variation. Therefore, the discussion patterns of teachers engaged in collaborative inquiry are an important aspect in the study of teachers' work together.

A Design to Develop Teachers' Noticing of Children's Mathematical Thinking in Collective Noticing

Although noticing children's mathematical thinking is fundamental to teaching that is responsive to students, researchers have documented that even though not all teachers demonstrate expertise in noticing (Dreher & Kuntze, 2015; Jacobs et al., 2010), their capacity to notice can be supported and developed (Goldsmith & Seago, 2011; Jacobs et al., 2010; van Es & Sherin, 2008). In order to provide opportunities for teachers to develop their capacity to notice children's mathematical thinking, the Responsive Teaching in Elementary Mathematics (RTEM) project has introduced teachers to research-based frameworks on children's mathematical thinking of rational numbers (Empson & Levi, 2011) and practices that elicit and build on children's mathematical thinking (Jacobs & Ambrose, 2008; Jacobs et al., 2011) through the examination of strategically selected student work. The student work was chosen to highlight characteristics of student thinking both in professional development sessions and in the self-facilitated school-based collaborative inquiry sessions.

Noticing children's mathematical thinking is a practice that teachers can learn. Teachers can develop their noticing through reflection and collaboration in a manner that is connected to their practice. Because teachers often work independently in their own classroom, in order to engage productively with one another the teachers must ensure their partner has enough background information to make sense of the discussion; Chamberlin (2005) found during these interactions that teachers were able to provide detailed descriptions of student thinking. Therefore, as teachers develop expertise in noticing children's mathematical thinking, noticing allows teachers to learn from their students and refine their practice.

When teachers engage in the collaborative inquiry sessions, they have an opportunity to share their noticing of children's mathematical thinking, making this practice visible to one another by verbalizing and critiquing the interpretations and decisions they make using the child's strategy as evidence of children's thinking.

Within the collaborative inquiry groups, teachers use student data in the form of written work and prior experiences to make sense of the strategy a child used, the understandings a child might have, and ways they might respond to support and extend those understandings. Teachers may discuss children's thinking in a variety of ways, such as: retelling an interaction they may have had with a child, describing what they understand about the strategy based on the child's written work, or by altering their voice to suggest what either a child or a teacher might say. In addition, because teachers may have different knowledge and experiences that are used to examine student work, noticing can be distributed across the group, potentially providing one another with new insights into student thinking. For this reason, examining how teachers interact together (e.g., the conversational moves teachers employ) to describe student work provides one way to better understand how teachers create opportunities to develop their capacity to notice children's mathematical thinking. Therefore, for the purposes of this study, I focus in particular on how teachers employ the components of noticing children's mathematical thinking to sustain conversations that are grounded in the details of students' strategies.

In the previous sections, I provided evidence to suggest how teachers can engage in the practice of noticing in self-facilitated collaborative inquiry, or collective noticing. As teacher noticing is not something teachers develop only by teaching in the classroom, Horn and Kane (2015) explored what role self-facilitated collaborations might serve for teachers who are not yet proficient in noticing children's mathematical thinking. My study aims to explore how teachers' patterns of interactions can open or constrain opportunities to sustain in the details of the student strategy through collective noticing. In the following chapter I discuss the methods I used to investigate the relationship between the teachers' interactions in collaborative inquiry and their quality of noticing children's mathematical thinking.

Chapter 3 : Description of the Collaborative Inquiry Tool

The Collaborative Inquiry Tool was an online tool that was designed to support teachers to work together in school-based teams to develop their expertise in noticing children's mathematical thinking. It was designed as a supplement to face-to-face workshop meetings in which the study teachers participated as part of the larger Responsive Teaching in Elementary Mathematics (RTEM) research project. In this chapter, I describe the different components of a collaborative inquiry module and provide examples of one of the modules, to familiarize readers with the tool.

As a member of the RTEM project team, I helped to design the web-based tool consisting of 13 collaborative inquiry sessions based on research on how children think about and solve problems. Figure 1 lists each collaborative inquiry module, its focus, and the problem(s) for which teachers were asked to collect students' work and bring to the session for discussion. The first session was designed as a practice session, in which teachers engaged during the first week of professional development. Each session was designed to engage teachers in face-to-face focused inquiry regarding children's thinking with key mathematical relationships through four main segments. These four segments included Prepare, Video or Written Work, Discuss Own Students, and Next Steps and are described in the following sections. In the following sections, I illustrate these segments using Module 5.

Module Number	Module Focus	Focal Problem
Practice	Children's thinking about whole-number multiplication	16 people are going to the theater. If each ticket costs $24, how much would it cost for 16 tickets? (7, $24) (36, $24)
1	Children's thinking about whole-number division: Early strategies	Coach Brown has 56 baseballs. 8 baseballs fit in a box. How many boxes can Coach Brown fill? (126, 10) (180, 12) There are 42 jellybeans in a bag. 7 children want to share them so that they each get the same amount. (122, 10) (250, 12)
2	Children's thinking about whole-number multiplication: Multi-digit numbers	Ms. Silver is planning to make cookies to give to her friends. She wants to give ______ cookies each, to ______ friends. How many cookies does she need to bake? (12, 10) (12,11) (12, 15) (15, 21) (32, 11) (2 dozen, 9) (11, 32) (7, 98) (18, 22)
3	Children's thinking about equal sharing: Early strategies	There are 11 pancakes for 4 kids to share equally. How much pancake does each kid get? (5, 8)
4	Children's thinking about equal sharing: Range of strategies	The zookeeper has 8 bananas to feed to the 6 monkeys. If she wants to use up all the bananas and give the same amount to each monkey, how much should she give each monkey? On a field trip to the museum 12 kids were given 16 churros to share equally. How much should each kid get?
5	Children's thinking about equal sharing: Early strategies	____ friends wanted to have some granola bars for a snack. They had _____ granola bars to share equally. How much granola bar can each friend have?

Figure 1. Collaborative Inquiry Module Information

6	Children's thinking about multiple groups problems: Early strategies	Divine has 12 giant chocolate bars to share with the kids on her soccer team. She wants to give each person 3/4 of a bar of chocolate. How many kids will she be able to give chocolate to before she runs out? Daniel's mom made 10 cheese sandwiches for snacks in Daniel's class. If each child gets 1/2 of a sandwich for a snack, how many children can have a snack?
7	Children's thinking about equal sharing: Equivalence relationships	_____children want to equally share _____ peanut butter sandwiches, with no leftovers. How much can each child have? (2, 6 1/2) (3, 7 1/2) (8, 5 2/4)
8	Children's thinking about multiple groups: Relational thinking strategies	Ms. Dolphin is thinking about buying _____ aquariums to put in the front office. Each aquarium holds _____ gallons of water. How many gallons of water will Ms. Dolphin need to fill all _____ aquariums? (5, 3 1/2) (3, 5 3/4) (7, 4 2/3)
9	Children's thinking about equal sharing: Using equations to represent key relationships	Maddy and her ____ friends want to share ___ sticks of licorice. How much licorice should each person get? (3 friends, 10) (2 friends, 8) [video] (7 friends, 3)
10	Children's thinking about unit fractions: Range of strategies	The zookeeper has ___ cups of frog food. His frogs eat ___ cup of food each day. How long can he feed the frogs before the food runs out? (3, 1/2) (4, 1/3) [student work] (4 3/8, 1/8)

Figure 1, cont. Collaborative Inquiry Module Information

11	Children's thinking about equations versus story problems	2 - 1/2 = _____ 4 - 1/3 = _____ 4 - 1/8 = _____ 4 - 1 1/6 = _____ You have ____ sandwiches. You eat ____ of a sandwich. How many sandwiches do you have left? (2, 1/2) (4, 1/3) [(4, 1/8) (4, 1 1/6)
12	Children's thinking about equations with unit fractions	1/2 + 1/2 + 1/2 = _____ 1/4 + 1/4 + 1/4 + 1/4 + 1/4 = _____ 4 5/6 = _____ x 1/6 _____ x 1/4 = 2 1/4

Figure 1, cont. Collaborative Inquiry Module Information

Prepare

In order to prepare for the collaborative inquiry session, teachers were asked to log in to the tool for the purposes of downloading a problem that was written to reflect the focus of the module. For example, the focus of Module 5 was for students to view and discuss early, or emergent, strategies for solving equal sharing problems.

The teachers could download the problem to copy for their students as either a portable document file (PDF) or a word processing file. The word processing file allowed teachers the opportunity to make adjustments in the problem context or number selection that were appropriate for their students. For example, one problem included the context of children equally sharing churros. One group of teachers decided churros would be unfamiliar to their students, so they decided to change the problem context to children sharing apples. Each problem also usually had a set of three number sets for students to solve. Some teachers only included one number set on the handout for all students to solve and would suggest additional number sets as students completed each problem.

For Module 5, teachers were asked to pose the following problem: *There are* ___ *pancakes for* ___ *kids to share equally. How much pancake does each kid get?* For this module, teachers (or students) were given a choice of two number sets, 4 share 11 and 8 share 5. Additional text informed the teacher they would later watch a video of a child solving 8 share 5.

The Prepare tab provided some suggestions for teachers to enact while their students solved the problem. For example, it was suggested they *unpack* the problem, or introduce to the problem context to the students and ensure the students understood what the problem was asking. In addition, the tool suggested teachers walk around and pose questions to students during the problem-solving task. In Module 5, it was suggested that teachers ask students how they decided to partition the pancakes. Lastly, the teachers were prompted to choose six pieces of student work to discuss with their colleagues.

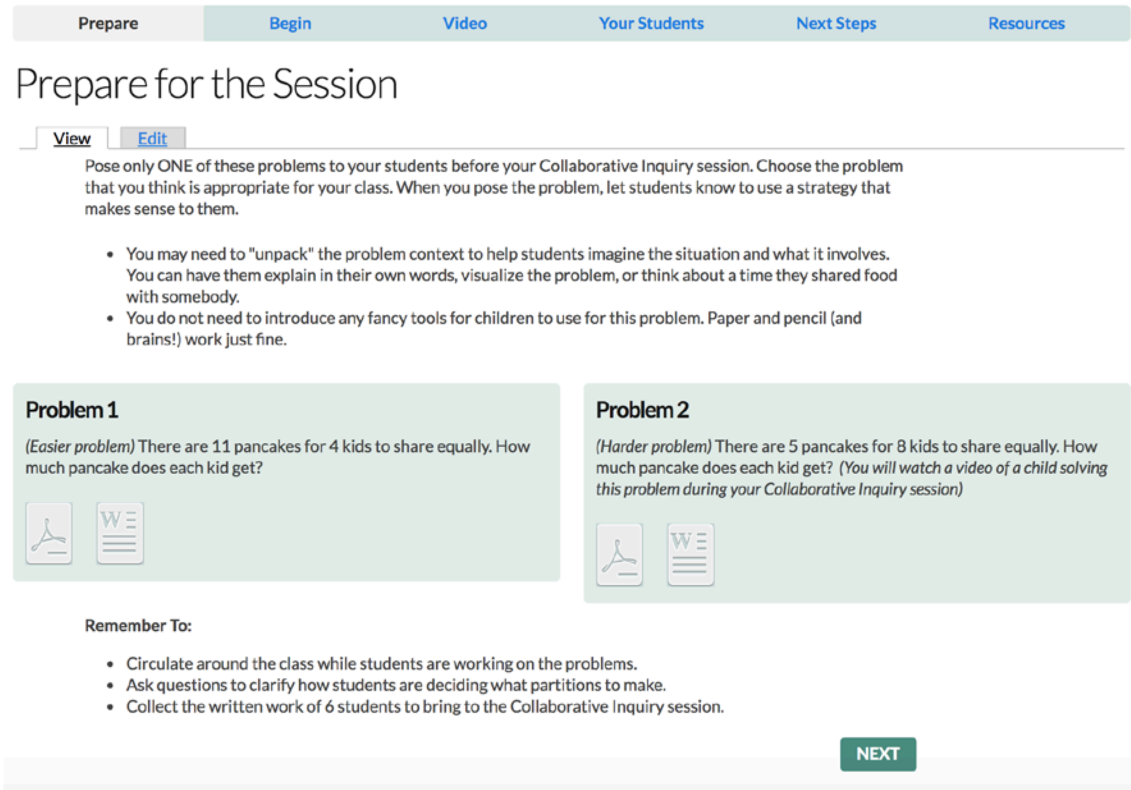

Figure 2. Example of Collaborative Inquiry Prepare page

Video or written work

To begin the session teachers were presented with a video of a student or written work from several students who had solved the same problem that the teachers posed to their students. The teachers were expected to view and discuss what they noticed about the mathematical thinking of the module's focal student or students.

In Module 5, the teachers had an opportunity to watch a fifth grader named Ryan solve the equal sharing problem 8 share 5 pancakes. In this video, teachers observed as Ryan solved the problem as a teacher asked questions to elicit Ryan's thinking about the problem (see Figure

3 for an image of how Ryan solved the problem and Transcript 1 to read the interaction between Ryan and the teacher). Through the interaction, Ryan solved the problem by partitioning five circles into eighths and finding an answer of 5/8. After the teacher posed questions about Ryan's strategy, she asked a follow-up problem to elicit how Ryan understood 5/8 as a quantity in comparison to 1/2. After a long wait time, when it seemed that Ryan was not going to be able to answer the problem, Ryan stated that 5/8 was 1/8 more than 1/2. Teachers were able to observe how the teacher's questions elicited and promoted Ryan's thinking, allowing him to persist in problem solving. Teachers then had an opportunity to discuss the details of Ryan's strategy and what they thought Ryan understood based on the details of his strategy. To see a video of this interaction, visit https://soe.uncg.edu/rtem/.

There are 5 pizzas for 8 kids to share equally. How much pizza could each kid get?

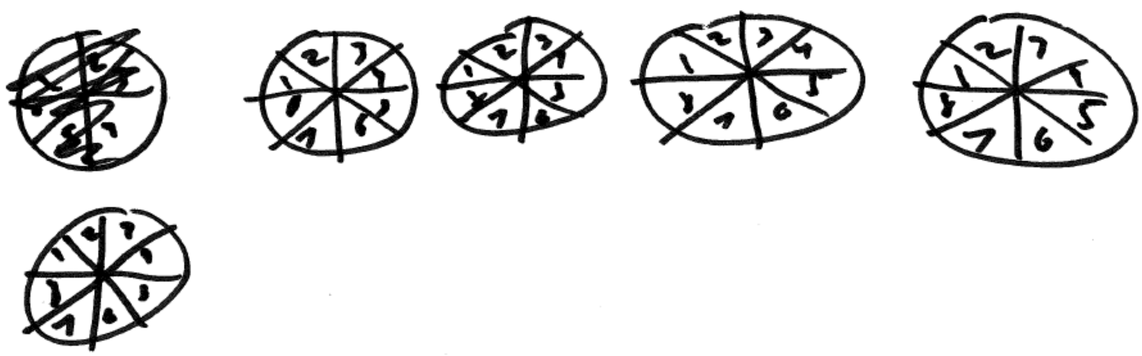

Figure 3. Image of Ryan's Strategy

Transcript 1: Ryan's Strategy

1. TEACHER: There are five pizzas for eight kids to share equally. How much pizza

could each kid get?

2. RYAN: (Starts drawing circles). Eight pizzas or five pizzas?

3. TEACHER: There are five pizzas for eight kids to share equally. How much pizza could each kid get? (Student draws 5 circles).

4. RYAN: Let's just see if each kid could get this (divides each of five circles into fourths and labels each fourth with a number 1–4). So 1/4 of one pizza (outlines the piece with 1 in the first pizza). Then another fourth which makes it 2/4, so you have 2/4 of two pizzas right now (outlines the pieces with 1 and 2 in the second pizza). Then you would have 3/4 of three pizzas (outlines the pieces with 1, 2, and 3 in third pizza). Then you would have one whole of a pizza (outlines the entire fourth pizza). Then you would have one whole and 1/4 out of all five pizzas (outlines the piece with 1 in the fifth pizza). One kid would have a whole pizza and 1/4 of a pizza.

5. TEACHER: And how many kids would be sharing if that happened?

6. RYAN: Eight kids.

7. TEACHER: So how do we know eight kids there or can you explain your picture?

8. RYAN: Oh. I messed up on that.

9. TEACHER: You want to try again?

10. RYAN: (Re-draws 5 circles). Okay, so five pizzas… (Partitions one circle into halves, then fourths, and then sixths but after numbering each piece realizes there were not enough pieces for eight kids. Marks out that circle and redraws the circle and splits into eighths and numbers the pieces 1–8). Okay. So five kids were sharing each pizza so –

11. TEACHER: (Clarifies) Eight kids were sharing five pizzas.

12. RYAN: Oh yeah sorry. Okay.

13. TEACHER: So, do you know how much each person is going to get already?

14. RYAN: No, I'm just dividing them equally. (Continues to split the remaining four circles into eight pieces each and number 1-8).

15. TEACHER: Okay.

16. RYAN: (Divides each circle into eighths and numbers each eighth in each pizza with a number 1–8). Okay. So, one kid would get 1/8 of a pizza (outlines the piece with 1 in the first pizza) and then another eighth of a pizza which would make it 2/8 of a pizza (outlines the pieces with 1 and 2 in the second pizza). And another eighth of a pizza would make it 3/8 of a pizza (outlines pieces with 1, 2, and 3 in the third pizza) and another eighth of a pizza would make it 4/8 of a pizza (outlines only the piece with 4 in the fourth pizza) and another eighth of a pizza would make it 5/8 of pizza (outlines only the piece with 5 in the fifth pizza). So each

kid would get 5/8 of each pizza.

17. TEACHER: Is that enough to have a whole pizza?

18. RYAN: No.

19. TEACHER: No. How do you know?

20. RYAN: Because it would have to be eight eighths to make one whole.

21. TEACHER: Is it enough to have 1/2 of a pizza?

22. RYAN: (10 second pause) No.

23. TEACHER: No. How do you know?

24. RYAN: Because if you are trying to make the fractions smaller, you can't condense five. You can condense four — no you can — because if you're con — okay, so if you're condensing, you'd get five. Okay, you condense four into — oh no, no you can't. That's right. You can't condense 5 into a smaller fraction.

25. TEACHER: So do you think they would have enough to have 1/2 a pizza or more than 1/2 or less than 1/2?

26. RYAN: Less than 1/2.

27. TEACHER: They would have less than 1/2 a pizza with 5/8 of a pizza, and how do you know?

28. RYAN: Because you can't — if you are trying to umm (20 second pause).

29. TEACHER: Is this a hard question?

30. RYAN: Yes.

31. TEACHER: Yeah, you've done great so far. So how much pizza is each person going to get?

32. RYAN: 5/8 of each pizza.

33. TEACHER: 5/8 of a pizza. Nice job.

34. RYAN: Actually, I know. They're going to get more than a 1/2 pizza because 4/8 would be one 1/2 then they basically have a — okay, which would make it 4/8 then they basically — which is a 1/2, then if it was 5/8 it'd be more than 1/2.

35. TEACHER: How much more than 1/2?

36. RYAN: One fraction.

37. TEACHER: One fraction? (Ryan nods). What would that fraction size be?

38. RYAN: (Brief pause) One. No. One. (7 second pause).

39. TEACHER: So, I heard you say 5/8 is more than 1/2 cause 4/8 is 1/2?

40. RYAN: Yes.

41. TEACHER: So how much more than 1/2 is 5/8?

42. RYAN: Oh, one fraction — one, one, uhh, one (brief pause)

43. TEACHER: What do you think?

44. RYAN: 1/2 or no…1/4?

45. TEACHER: 1/4 more than 5/8? Why?

46. RYAN: No, 1/8 because if you added 1/8 to 4/8 it would make it 5/8.

47. TEACHER: Nice job hanging in there with that one.

Explore your Students' Work

After teachers discussed the focal student or students, they were asked to review the written work of their own students that they brought to the meeting. Teachers were prompted to discuss what they noticed about the mathematical thinking of at least one student from each of their classrooms. Specifically, teachers were asked to describe each student's strategy in detail and discuss the potential understandings of the student as revealed by the strategy. To help facilitate this discussion, the tab included a written description of some things that the project team noticed about the focal student's mathematical thinking. These descriptions were meant to be illustrative and did not include all of the possible ideas that could have been noticed.

For example, in Module 5, the project team highlighted Ryan's emergent understanding of 5/8 as a sum of 1/2 and 1/8. Teachers were able to interpret Ryan's understanding through the use of the questions the teacher in the video posed to Ryan in order to elicit his understanding of five-eighths as greater or less than one-half, and a follow-up question of how much greater five-eighths was than one-half. This section of the collaborative inquiry tool was the focus of the dissertation.

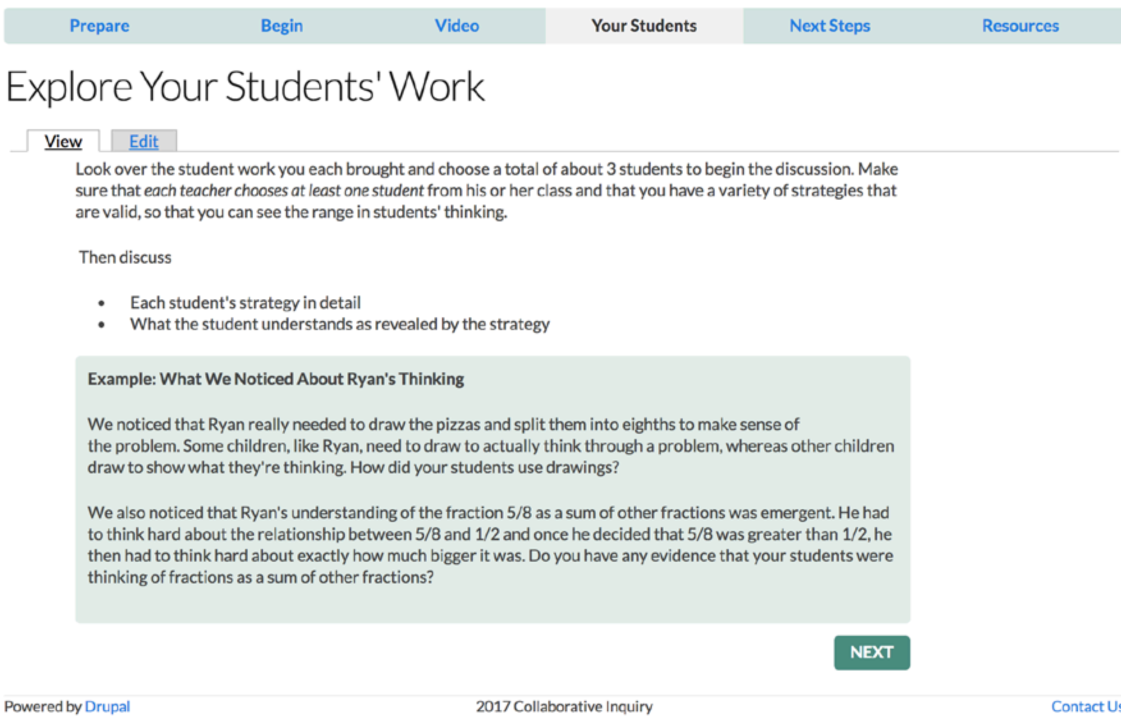

Figure 4. Example of Collaborative Inquiry Explore Your Students' Work page

Next steps for own students

Teachers were then asked to use what they learned about one student's mathematical thinking to design a follow-up problem that they could pose within the following week. The project team provided sample follow-up problems with articulated reasoning based on what was noticed about the case study's mathematical thinking to support teachers in this task.

For Module 5, the project team suggested posing another equal sharing problem with fraction amounts that may be more familiar than eighths to Ryan, such as fourths, to help strengthen his understanding of a fraction as a sum of other fractions. One suggested problem was *Four children shared 3 same-sized sub sandwiches so that each person got the same amount. How much did each person get?* Ryan most likely would solve this problem by partitioning each sub sandwich into fourths and then adding three groups of one-fourth for a total of three-fourths. Ryan's understanding of the equivalence between one-half and one-fourth could then be elicited.

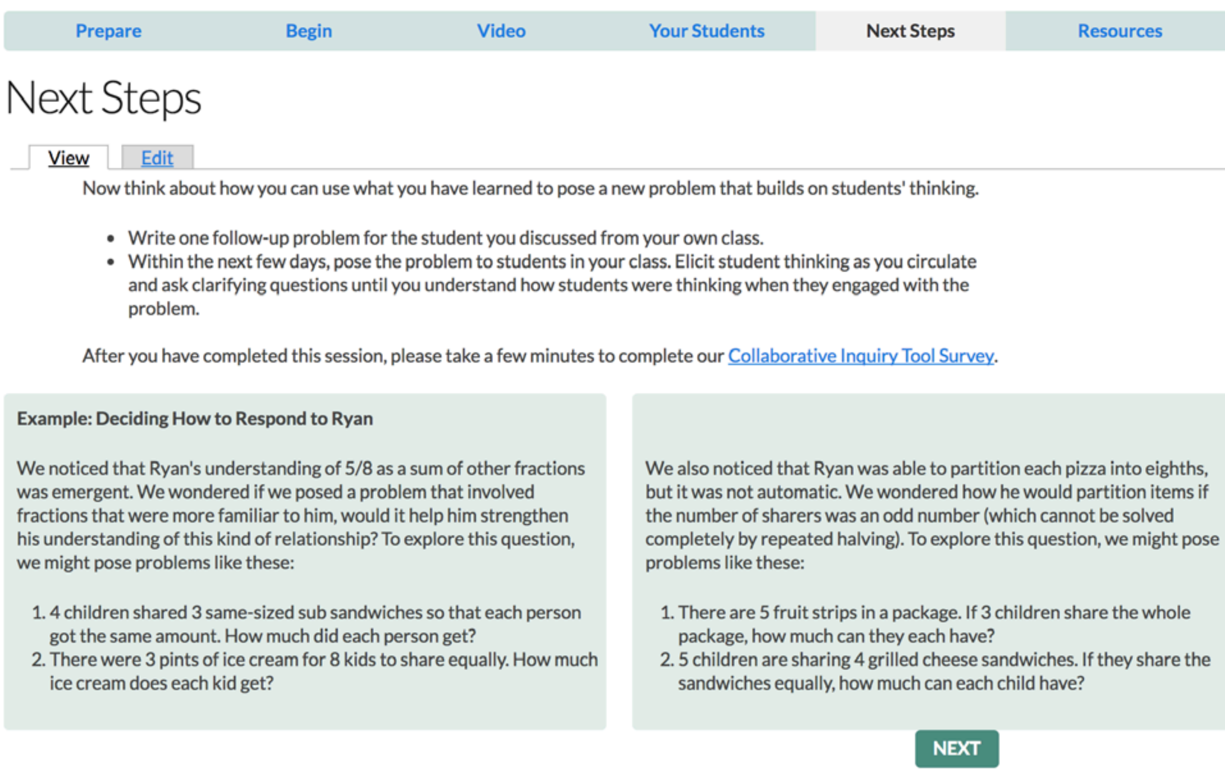

Figure 5. Example of Collaborative Inquiry Prepare page

Chapter 4 : Methodology

In this instrumental case study (Stake, 2005) of teachers' collaborative inquiry sessions, I explored how 13 groups of 3rd–5th grade teachers worked together to develop their capacity to notice children's mathematical thinking. Instrumental case studies "provide insight into [the] issue" (Stake, 2005, p. 445) of the opportunities afforded to teachers participating in collective noticing in school-based collaborative inquiry groups. Using qualitative research and discourse analysis techniques I studied the relationship between teachers' interactions in collaborative inquiry and their engagement in the practice of noticing to provide insight on the opportunities created by teachers participating in collective noticing in school-based collaborative-inquiry groups. This study examined the following question:

1. How do teachers collectively engage in the practice of noticing the mathematical thinking of children when participating in self-facilitated collaborative inquiry?
   1.1. What is the quality of teachers' collective noticing when discussing student work together?
   1.2. What are the teachers' patterns of interaction in collective noticing when discussing student work?

Study Participants

Participants who received an invitation for data collection came from a larger professional development design study (Empson & Jacobs, 2012), involving 92 3rd–5th grade teachers and instructional facilitators who worked with classroom teachers. As a part of that study, teachers attended up to 150 hours of face-to-face professional development focused on teaching that is responsive to children's fraction thinking over three years (see Figure 6). Each year of the professional development, teachers attended one weeklong session during the summer, two consecutive follow-up days in the fall, and two consecutive follow-up days in the spring. As a required component of the professional development, teachers were expected to form school-based teams to engage in four collaborative inquiry sessions each year, outside of the scheduled professional development days. Teachers' participation in these collaborative inquiry sessions provided the data for this study.

The participants that made up the data set for the final analysis, after data collection and reduction, consisted of 30 3rd–5th grade teachers and one special education teacher. Thirteen teachers participated in Cohort A, 13 participated in the Cohort B, and four participated in the Cohort C. The teachers entered in to the professional development with a range of teaching experience from one to 34 years (with an average of 11 years), and roughly one-fifth reported participating in at least one year of professional development focused on children's mathematical thinking with whole numbers previous to the study.

The participants were employed in one of three neighboring school districts in the southern region of the United States. The researchers selected the districts because the administrators endorsed the professional development and instruction that was responsive to children's mathematical thinking; however, the districts varied in their instructional contexts and histories. Two districts had long histories of supporting their teachers in learning about children's thinking to inform instruction, while the third school district had recently begun to focus on teaching that is responsive to children.

	13-14	14-15	15-16	16-17
Cohort A	Year 1 PD (50 hours)	Year 2 PD (50 hours)	Year 3 PD (50 hours)	
Cohort B		Year 1 PD (50 hours)	Year 2 PD (50 hours)	Year 3 PD (50 hours)
Cohort C			Year 1 PD (50 hours)	

Figure 6. Cohort by year of professional development

Data Collection

A total of 12 collaborative inquiry modules were designed for teachers to complete over three years outside of professional development sessions. Of these, 11 focused on fractions and served as the basis for my study. For each of the 11 collaborative inquiry modules focused on fractions, I asked teacher groups to audio record their discussion and send copies or images of all student work that was discussed in the session during the academic years between 2013 and 2017. In an effort to collect a variety of teacher interactions in these collaborative inquiry groups I asked every teacher group to audio record at least one session per academic year, resulting in approximately 135 data collection requests. For each collaborative inquiry module I contacted at least one teacher group from each district to record their session. I asked teacher groups to record their discussions using an accessible audio device and make copies of the student work they discussed during the session. I offered to send a digital audio recorder if a device was not accessible. See Figure 7 for group sessions considered in

Audio Recording the Sessions. Most teachers chose to digitally record using the voice recorder on either their smart phone or tablet device. A few groups broke up their recording to keep the file size small. Two teacher groups set up a video recorder to capture their discussion, and one teacher group tried a variety of accessible recording devices. In order to collect the data remotely, I set up a shared cloud-service (Box) folder for each group to upload their audio recordings and scanned images of the student work; however, most teachers emailed the data from their sessions or shared their files using one a cloud-based service set up by the district. In addition, teachers brought data to the follow-up professional development sessions or mailed copies of their student work using a pre-paid envelope. The audio-recorded observations in combination with copies of student work form the basis for the dataset for this dissertation in order to explore the teachers' noticing practices. A summary of the analyzable sessions can be found in Figure 7.

Group Characteristics		Session Information		
Group Number	Teachers (Grade)	Module Number	Number of Minutes Discussing own students	Number of Episodes Identified
1	Jill (3rd), Melissa (4th), & Shelby (3rd)	1	15:19	10
2	April (3rd), Ronda (3rd), & Sally (5th)	1	10:35	8
3	Marley (3rd) & Susie (3rd)	1	07:01	4
4	Stella (5th) & Kim (4th/5th)	1	05:34	4
1	Jill (3rd), Melissa (4th), & Shelby (3rd)	2	14:18	8
5	Lori (4th) & Maddy (Special Ed)	2	06:51	5
6	Kiara (3rd), Lynette (3rd), & Thea (3rd)	2	08:08	7
4	Stella (5th) & Kim (4th/5th)	2	05:05	5
1	Jill (3rd), Melissa (4th), & Shelby (3rd)	3	16:27	16
7	Janice (4th) & Molly (4th)	4	18:36	8
1	Jill (3rd), Melissa (4th), & Shelby (3rd)	4	13:29	12
5	Lori (4th) & Maddy (Special Ed)	4	06:35	6
1	Jill (3rd), Melissa (4th), & Shelby (3rd)	5	5:30	7
8	Debra (5th) & Claudia (4th)	5	29:52	14
9	Gladys (4th) & Todd (3rd)	5	02:57	3
1	Jill (3rd) & Melissa (4th)	6	13:26	7
1	Jill (3rd) & Melissa (4th)	7	19:40	8
10	Erica (5th) & Sydney (3rd)	7	06:04	6

Figure 7. Final Data Collection Table for Analysis

7	Janice (4th) & Molly (4th)	8	27:48	14
1	Jill (3rd) & Melissa (4th)	8	12:22	6
1	Jill (3rd) & Melissa (4th)	9	11:45	7
11	April (4th) & Emily (5th)	9	8:31	11
12	Dennis (5th), Leti (3rd), & Silvia (4th)	9	18:55	13
5	Lori (4th) & Maddy (Special Ed)	9	03:52	4
6	Kiara (3rd), Lynette (3rd), & Thea (3rd)	9	05:16	3
1	Jill (3rd) & Melissa (4th)	10	5:07	6
5	Lori (4th) & Maddy (Special Ed)	10	03:29	8
13	Daniel (4th) & Sage (4th)	11	12:14	5
14	Eleanor (4th) & Shauna (4th)	11	17:39	5

Figure 7, cont. Final Data Collection Table for Analysis

Data Reduction

Over the four years of data collection, 36 unique teacher groups submitted audio recordings from a total of 112 collaborative inquiry sessions. Not all of the data collected was usable for analysis and the dataset needed to be reduced. For example, teachers' audio recording devices occasionally failed or teachers may have forgotten to collect their student work to submit before giving it back to the students. After data reduction was completed, the initial data set was reduced to 29 analyzable sessions from 14 teacher groups for a total of 11 hours and 32 minutes of audio data. To be included as analyzable, data from a collaborative inquiry session had to meet the following criteria:

- All group members were participants in the same professional development cohort in the larger RTEM project.

- Students' written work was submitted.
- The session focused on children's fraction reasoning.
- The recording was audible and complete (i.e., included the entirety of the teachers' session together).

The reasoning behind these criteria is explained in the following sections.

Group members are teacher participants in the RTEM study. During recruitment for the RTEM study, teachers were asked to attend the professional development with at least one teacher from their school. However, due to a variety of reasons, including attrition, some teachers did not have a partner teacher. In order to complete the collaborative inquiry sessions, some teachers chose to work with other teacher participants from nearby schools, while others created school-based collaborations with teachers that did not attend the professional development to engage in the sessions with them. For the purposes of my study, only sessions in which all teachers participated in the research project were considered. Reducing the data in this way allowed for the assumption that all teachers participating in the collaborative inquiry had received the same research-based frameworks on children's mathematical thinking and instructional practices.

Submitted written student work. Additionally, as my analysis explored what teachers noticed about children's mathematical thinking, only sessions in which teachers submitted scans of the student work that was discussed was considered. This allowed me to visually follow and confirm the mathematical details of a child's strategy that the teachers highlighted in their discussions.

Focus on fraction reasoning. As the professional development introduced research-based frameworks on fraction thinking, teacher sessions for the first two modules, which focused on children's thinking of whole number multiplication and division, were not considered for this analysis. See Figure 7 for a complete description of the collaborative inquiry modules that were analyzed.

Audible and complete recordings. Each collaborative inquiry session was reviewed to determine whether the recording included the following three components: discussion of sample video or student work provided, discussion of own students, and discussion of deciding how to respond. Sessions in which teachers did not discuss their own student work were not considered for further analysis. In addition, these audio recordings were considered for audibility, such that each teacher voice is mostly heard and understood during these discussions.

Identifying episodes of discussing children's written work. After reviewing each of the submitted sessions to ensure the four above criteria were met, I uploaded 31 audio recordings and the corresponding scanned images of student's written work into MAXQDA (Version 18.0.5, VERBI Software, 2018), a computer-based qualitative data analysis program designed for audio, video, and portable document files to prepare for data analysis. I then reviewed the data to parse it into the unit of analysis, or episodes when teachers discussed the written work of their students.

Episodes considered for analysis consisted of instances in which the teachers had an opportunity to discuss what they noticed about individual children. While the sessions were designed to provide opportunities for teachers to engage in noticing children's mathematical thinking, no facilitators were in attendance to keep the discussion focused to this task.

Occasionally, as teachers talked about an individual child, the discussion shifted to a more general discussion about the class or another student. However, if the partner teacher(s) did not take up the shift or the discussion returned back to the original child, the entire excerpt was counted as one unit. This unit was used to strategically reduce the data set to identify episodes when teachers had an opportunity to engage in the practice of noticing children's mathematical thinking. I then matched the corresponding piece (or pieces) of student work with the episode by listening for details that unambiguously connected the episode with the piece of student work. When the student work was matched to the appropriate instance, the unit was identified as an *episode of discussing children's written work*. After reviewing the 31 audio recordings, 220 episodes of discussing children's work from 29 sessions were considered for the final analysis.

To understand how teachers worked together to engage in the practice of collective noticing, I coded these sessions for two aspects: the quality of the noticing of children's mathematical thinking and the patterns of the teachers' interactions. The substance was analyzed in terms of the teachers' noticing of children's mathematical thinking. The form was analyzed in terms of the interactional patterns of teachers' discussions when noticing children's mathematical thinking. To aid in this analysis, episodes of discussing children's written work were transcribed and coded in MAXQDA.

How do teachers collectively engage in the practice of noticing the mathematical thinking of children when participating in self-facilitated collaborative inquiry?		
Analysis phase	Research subquestion	Data analyzed
Data reduction		Audio corpus of submitted collaborative inquiry sessions
Phase one: Coding the quality of teachers' collective noticing.	What is the quality of teachers' collective noticing when discussing student work together?	Episodes of discussing children's written work
Phase two: Coding the patterns of teachers' interactions during collective noticing.	What are the teachers' patterns of interaction in collective noticing when discussing student work?	Conversational turns within episodes of discussing children's written work
Phase three: Generative Collective Noticing	How do teachers collectively engage in the practice of noticing the mathematical thinking of children when participating in self-facilitated collaborative inquiry?	Phase 1 and Phase 2 codes for episodes of discussing children's written work

Figure 8. Phases of data analysis and related research questions

Data Analysis

The data analysis consisted of three phases. In phase one I analyzed the quality of what teachers noticed about the mathematical thinking of individual students. In phase two I analyzed the quality of the conversational interactions among teachers when they discussed the written work of their students. In phase three I analyzed the relationship between what teachers noticed and their conversational interactions about the mathematical thinking of their students.

Phase one: Coding the quality of teachers' collective noticing. In this analysis phase, episodes of discussing a child's written work were analyzed, using an adaptation from Jacobs et al. (2010) to determine the ways teachers' discussions were grounded in the details of the student strategy. For each episode of discussing a child's written work, I created a memo to identify what teachers noticed about the child's mathematical thinking and included an image of the relevant student work.

Reading the transcripts I was interested holistically in three characteristics of teacher noticing: (a) in what ways the teachers' discussion was grounded in the details of the student's strategy, (b) the potential student understandings teacher identified as revealed by stated details, and (c) the ways they described how they might respond to the student. While reviewing the transcripts, I not only identified what teachers noticed about the student's strategy, but also used the teacher descriptions to reconstruct the student strategy. This analysis does not make claims about either an individual or group of teachers' capacity to notice children's mathematical thinking, but rather how the group explicitly verbalized how their noticing was grounded in the details of the child's strategy. Therefore, I coded each episode as one of three main categories: (1) lack of evidence; (2) limited evidence; and (3) robust evidence of collective noticing children's mathematical thinking. A summary of these codes can be found in Figure 9. These characterizations were based on how grounded the discussion was in the details of the student strategy.

Lack	Limited	Robust
described minimal mathematical details and no evidence of child's reasoning	described some of mathematical details in ways that are **isolated** from the child's reasoning	described the majority of the mathematical details in ways that were **connected** to the child's reasoning

Figure 9. Characterization of the Quality of Noticing

When teachers discussed a connection between two students, I coded the episode as linked but distinct episodes. This identification helped me consider how teachers may have implicitly or explicitly used details from the first strategy when considering the second strategy, potentially bumping up the identified code. Additionally, when teachers discussed details of more than one strategy within an episode, I considered the discussion for each strategy and identified and holistically coded the entire episode using the highest identified code. It is important to note that the purpose of this coding is not to make claims about any teacher's individual capacity to notice, but rather identify what the teachers made visible to one another about what they noticed and how each teacher explicitly connected that noticing to the student's strategy. Occasionally students used mental strategies to solve the problem and did not represent their thinking. I considered how teachers made claims about how the child may have solved the problem when determining how to code the episode.

Robust evidence of collective noticing of children's mathematical thinking. Episodes that were coded as robust evidence of noticing children's mathematical thinking generally described the majority of the mathematical details and connected those details to the problem context and the child's mathematical reasoning. When reviewing the transcript, most of the strategy could be recreated indicating the details were described in a way that was grounded in the student's written strategy.

Limited evidence of collective noticing of children's mathematical thinking. Episodes that were coded as limited evidence of noticing children's mathematical thinking generally described some mathematical details and engaged in the child's mathematical reasoning, but often in ways that were isolated from one another. Episodes that were coded as limited often had incomplete or vague descriptions, which made it difficult to reconstruct the student strategy only using the transcript from the episode. This was an important distinction because it indicated that the teachers had not verbalized important strategy details or had not discussed the details in a way that had considered the child's process.

In addition to describing the details in incomplete ways, teachers may also have described the details without making a connection to the context of the problem. While the strategy may have been recreated, there was little connection to how the child may have used the context of the story problem to solve the problem.

Lack of evidence of collective noticing of children's mathematical thinking. Episodes that were coded as a lack of evidence of noticing children's mathematical thinking generally described minimal mathematical details and often provided no evidence of engaging with the child's reasoning. While reading the transcript only, I was not able to recreate the student's strategy. If the student used a mental strategy and the teacher did not indicate engagement with the child's reasoning by either making a claim or wondering about how the child solved the problem, the episode was also considered as lacking evidence of engaging with the child's reasoning.

In addition, I identified episodes where teachers described how they talked to students prior to the student answering the question, or after the student incorrectly answered the problem. With these episodes I created a rule to determine how to categorize this special case. I decided that when a teacher described posing a set of directive questions to help the student achieve a correct answer was a lack of noticing children's mathematical thinking. I made this decision because interactions that can interrupt the child's strategy or are a series of closed questions can be considered as moves that take over the student's thinking (Jacobs et al., 2014), rather than elicit and build from student thinking. Therefore it was difficult to determine what the teacher noticed about the student's mathematical thinking. However, if teachers described ways they asked questions to help the student better understand the story problem and then walked away, I continued with my coding scheme.

Open coding within the quality of collective noticing. While I reviewed the episode transcripts to code the quality of noticing, I began to consider additional codes that could contribute to the teachers' opportunities to notice. For example, many teachers began to discuss an interaction they had with the student, so I created two additional codes to capture if the teacher talked to the student one-on-one during class, or discussed the student's strategy with the class. In addition to coding for these interactions, I also recorded how the teacher decided to respond in each episode memo, noting questions or prompts the teacher reported posing. As additional codes were created, I systematically reviewed previous episodes to confirm existence.

Coding reliability for the quality of collective noticing. After each episode was coded for the quality of noticing, I printed out, sorted, and read through each episode memo. The purpose of the sort was to ensure I had not drifted in my coding, and each episode represented the identified quality of noticing. During this sort, some episodes were moved into another category and the additional substance open-codes were verified. I then asked another project team member to independently review 20%, or 44, of the episodes using the code descriptions, the episode transcript, and an image of the student's written work. After coding was complete, I calculated the percentage agreement as 50%. We then compared our coding and discussed any differences in the coding, coming to an agreement on all but two of the episodes.

Phase two: Coding the patterns of teachers' interactions during collective noticing. After coding the episodes of discussing children's written work for the quality of teachers' noticing, I reviewed and coded the identified episodes to capture the patterns of teachers' interactions that may open up teachers' opportunities to collectively notice children's mathematical thinking (Little, 2003). I integrated speech act theory and conversation analysis techniques in order to consider how the teachers' interactions contributed to the quality of their noticing. However, as my study explored teachers' collective noticing, I was most interested in identifying sequences of turns in which both teachers worked together to notice children's mathematical thinking, and in particular how a partner teacher takes up and expands on a detail or an interpretation.

To consider how the teachers made sense of the student's strategy together, I read through all transcribed episodes of student work and identified adjacency pairs (Schegloff, 2007), or a sequence of two proximate and related turns produced by two different teachers. The second turn of the adjacency pair does not necessarily immediately follow the first; however, the two turns must be linked or the second turn is a contribution to the initial turn. In addition, adjacency pairs can span more than two turns. For the purposes of my analysis, I considered an adjacency pair to be a sequence of turns related to noticing the student's mathematical thinking, or conversational interactions related to noticing children's mathematical thinking.

Characteristics of conversational interactions related to noticing children's mathematical thinking. To create the code system for the conversational interactions related to noticing children's mathematical thinking, I considered common ways the teachers took conversational turns. For example, a teacher might describe the student's strategy, and the partner teacher might add an additional description; or a teacher might ask a question about the student's strategy, and the partner teacher might respond. I then selected approximately 10% of the episodes coded within each characterized quality of noticing (10 episodes coded as robust, 10 episodes as limited, and six episodes coded as lack of evidence of noticing children's mathematical thinking). These episodes were selected in an attempt to capture the variety of turns that may exist among the quality of noticing children's mathematical thinking.

I reviewed each transcript looking for instances of an interaction, or a conversational turn where the partner teacher began to speak. If the partner teacher's turn only consisted of sounds that demonstrated listening or agreement, or one-word responses (such as "yeah," "okay," or "right"), a conversational interaction related to noticing children's mathematical thinking was not identified. While these responses may be important to communicate listening to the speaker, I was interested in identifying instances where the teachers co-constructed the noticing of children's mathematical thinking. In addition, Mercer (2013) posits the partner teacher potentially had an opportunity to develop within this interaction; however, the evidence to make this claim is limited because the teacher has not indicated what they are taking away from the interaction.

Rereading the sequence of turns (what was stated before and after the partner teacher's turn), I considered what idea the teachers were engaging with as it related to the child's written strategy and how the idea was being taken up within the turn. After identifying a conversational turn, I used open descriptive coding (Miles, Huberman, & Saldaña, 2014) leaning on speech act theory to consider the function of each turn. For example, if the teacher in the initial turn was describing the details of the student strategy, and the teacher in the second turn clarified the description, I named this conversational interaction as Describe-Clarify. After identifying the sequence of speech acts, I then created sub-codes to describe the function of each conversational interaction, or contribution, related to noticing children's mathematical thinking.

For the next round of coding I selected the episodes where teachers described an interaction they had with the student while he or she was solving the problem. I chose to focus on these episodes to ensure I captured the types of interactions the pair of teachers had when one teacher shared an account, or story, from the classroom. When coding these episodes, I identified speech act units where the teacher began to retell the interaction and coded these units as *accounts.* I then looked for the existence of conversational turns within these episodes.

After coding these initial sets, I began to code the remaining episodes in 10% increments, selecting episodes from each of the identified quality of noticing categories. As I identified new conversational turns, and after each round, I would review the identified codes to verify whether the selected units characterized the category or if the units needed to be recoded.

Levels of contributions related to noticing children's mathematical thinking. After coding for the conversational interactions related to noticing children's mathematical thinking, I defined three levels of contributions — no contribution, low contribution, and high contribution to consider how the interactions contributed to the co-construction of noticing. This continuum of sharing knowledge was considered leaning on two of Mercer's (2000) three ways of children's talk: *cumulative* and *exploratory*. Mercer described *cumulative talk* as individuals uncritically building on ideas to construct knowledge, and *exploratory talk* as individuals critically building on ideas through alternative suggestions and justification of their thinking. Mercer (2004) stated these "three types of talk were not devised to be used as the basis of a coding scheme" (p. 146), but to allow researchers to make sense of the different ways people interact with one another. I propose the interpretation of cumulative talk in two ways, which could influence the level of contribution.

In one interpretation, I consider cumulative talk as teachers' conversational turns add to the conversation in a way that continues, but does not shift the idea unit. I contrast this with Mercer's exploratory talk, where teachers take up an idea, but then shift the idea unit in a way that the initial teacher did not intend or consider. For my analysis, I considered both of these types of uptakes to be a higher level, as both teachers are contributing to the shape of the conversation, or working together to make sense of children's thinking. See Figure 10 for more examples.

I also consider a second interpretation of cumulative talk where teachers reiterate or demonstrate agreements in one another. While the second teacher is showing engagement, these interactions do not make new noticing children's mathematical thinking contributions to the conversation in a way that shapes the group's professional noticing. For my analysis I consider this interaction to be a low form of contribution.

Characterizing episodes as levels of conversational interactions related to noticing

children's mathematical thinking. To prepare for the third phase of the analysis, I characterized each episode with a contribution related to noticing children's mathematical thinking considering the varying forms of contributions. If an episode contained at least two high contributions, it was characterized as an episode that opened up the teachers' opportunity to jointly construct a child's mathematical thinking together. If an episode consisted of only one high contribution, low contributions, or no contributions related to noticing, it was characterized as an episode that opened up the teachers' opportunity to independently construct a child's mathematical thinking.

Code/Subcode	Definition/Example
High Contribution Turn Units	A turn or sequence of turns that function to engage at least two teachers in noticing children's mathematical thinking
Elaboration	T1: So, I didn't get over to question him but I would have gone back and said, okay can you show me where your sandwiches are instead of your—are these your sandwiches? Okay, so this is how much of a sandwich. So, this is one sandwich. So, do you have 10 sandwiches or do you have 10 pieces or halves? T2: And maybe just the question, show me your—how many sandwiches are in our story, 10. Show me your 10 sandwiches. Maybe he could figure it out, right then.
Counterclaim	T2: I know, I'm wondering if from here I can infer that the four-fourths equals a whole. I really feel that way from the way she shared out those first two pieces. That she's got that understanding, it's just a matter of notation that messed her up, right? T1: But then she just-, from there on she was no longer thinking three-fourths. She was just taking a half from each and making whole...There is no way to—she is not showing that she understands even if there are four fourths in a whole T2: No, she's not.
Claim	T1: I pulled this apart from some of the other three-fourths in the same way that you just commented that it shows that the four-fourths equals a whole and that you can pull out that one-fourth leftover from your first share and add it to the two-fourths of the next bar. And not a single fraction written on that. T2: He understands that there's four equal parts but there's no, yeah, no fraction is written.
Description	T1: I guess she was thinking six equal parts down there. Instead of thinking of a third. T2: Oh, she was, yes. These are the two [thirds].

Figure 10. Classification of Conversational Interactions Related to Noticing

Deciding how to respond	T1: I don't know if she would know that that's the same as one and one-fourth. T2: So, it would be an interesting question to ask her. Since she put four-fourths plus one-fourth, if she knows that four-fourths was the whole.
Low Contribution Turn Units	A turn or sequence of turns that allow one teacher to engage in noticing children's mathematical thinking
Repetition	T1: So each person gets one, two pancakes and then we split the three pancakes into fourths. T2: I think we should each get two pancakes and split three pancakes into…. Okay, good.
Agreement	T1: And he took one away so three-fourths were left over T2: Yeah, I think so too.
Clarification	T1: Yeah I wanna know how she does. I'm curious to know why did she do six-fourths. Why did she combine two of them but not… T2: And then not the last one? T1: Yeah.
No Contribution Turn Units	A lack of conversational turns related to noticing children's mathematical thinking.

Figure 10, cont. Classification of Conversational Interactions Related to Noticing

Phase three: Generative Collective Noticing. After the quality of noticing and teacher interactions were identified for each of the episodes of noticing children's written work, I explored how the patterns of interactions were associated with the quality of noticing children's mathematical thinking in collective noticing. I merged the two phases and exported and matched the identified quality of noticing and the conversational turns. I then characterized each episode as lack-no contributions, limited-no contributions, robust-no contributions, lack-low contributions, limited-low contributions, robust-low contributions, lack-high contributions, limited-high contributions, and robust-high contributions. I then selected a session to demonstrate how the characterized dimensions may have opened up opportunities for the teachers to engage in collective noticing that is generative, or continually enhanced through participation.

Summary of Methodology

To explore the characteristics of how teachers engaged in self-facilitated collective noticing I analyzed 220 episodes of discussions of student work. During the analysis I considered two dimensions of the teachers' discussions: (a) the quality of their collective noticing as evidenced by discussions that were grounded in the details of students' strategies, and (b) the patterns of interactions that opened up opportunities to notice collectively. I then compared these two dimensions to consider how teachers sustained in conversations that were grounded in the details of the students' strategies. I present my findings from this analysis in the next chapter.

Chapter 5 : Characterizing Collective Noticing of Children's Mathematical Thinking in Collaborative Inquiry

This chapter presents the results from my analysis of what and how teachers *collectively noticed* children's mathematical thinking. Collective noticing is operationalized as a group's capacity to ground its conversations in and articulate the details of student strategies. I analyzed episodes of teachers discussing written student work to characterize the quality of teachers' collective noticing and their interactional patterns. I used these characteristics to consider how teachers' self-facilitated discussions opened up opportunities for collective noticing.

I begin with a descriptive overview of the session and episode characteristics. While I analyzed the episodes as isolated units to consider the variation, episodes were embedded within the sessions. I then present my findings for the quality of the collective noticing of children's mathematical thinking, using an analytic framework adapted from Jacobs et al. (2010). Following this, I present how teachers' patterns of interactions allowed the teachers to construct student thinking either jointly or independently. Lastly, I present one group's session to show how both the quality and the patterns of interactions potentially opened opportunities for the two teachers to jointly construct collective noticing.

Descriptive Statistics

For my analysis, I reviewed 29 sessions of teachers engaged in collective noticing and analyzed, in particular, teachers' discussions about the mathematical thinking of their own students, as represented by the written work for a problem. Students in each classroom solved the same or a similar problem (e.g., varying in the number choices). The protocol asked every teacher to schedule about 45 minutes to discuss the provided *Video or Written Work of a child solving a problem, Explore [their] own students' work*, and discuss *Next steps.* The protocol suggested each teacher bring six pieces of student work for the problem.

Sessions. Before teachers began to discuss the student work from their own classrooms, they were prompted to select three pieces of student work, at least one from each classroom, which demonstrated a range of mathematical thinking within the set. Across the sessions I identified a range of three to 16 pieces of student work discussed per session, and teachers sustained these discussions from 3 minutes up to 28 minutes. On average, teachers discussed 7.5 pieces of student work per session (see 1), for approximately 11.5 minutes per session. On average, each teacher discussed three pieces of student work in the session, the total number for the session recommended by the protocol. As teachers discussed so many pieces of student work, this might suggest they were interested in looking through a range of student work from their own classrooms in order to have a better picture of how their own students were engaging with the mathematics. However, discussing more than three pieces of student work in the suggested time frame could potentially limit opportunities to engage in noticing the mathematical thinking of one student, as teachers may be choosing breadth over depth.

Table 1

Session Characteristics

Total Sessions	Duration Range of Discussing Own Student Work	Average Duration of Discussing Own Student Work	Range of Episodes per Session	Average Episodes per session
29	00:02:57 – 00:27:48	00:11:28	3–16	7.5

Episodes. Across the sessions, I identified and analyzed 220 episodes of discussing children's written work. An episode is an instance in which the teachers had an opportunity to discuss what they noticed about an individual child. For this study, I chose to consider the episodes in isolation as one way to consider variation, although I noted if the teachers named an explicit connection across episodes. The length of each episode ranged from 5.5 seconds to 5.5 minutes (see Table 2), and the average episode was 1 minute and 6 seconds. This suggests the teachers' capacity to sustain discussion about the mathematical thinking was varied, from describing the student's answer to describing all of the mathematical details in the student's strategy. However, while 29 episodes were longer than 2 minutes, for some of these episodes the conversation shifted to a secondary discussion before returning to the episode's focal student, suggesting the longer durations may be an overrepresentation of teachers' capacity to sustain in conversations grounded in the details of student strategies.

Table 2

Episode Characteristics

Total Episodes	Duration Range	Average Duration
220	00:05.6-05:29	01:06

Quality of Noticing Children's Mathematical Thinking

In this section, I address the first sub-question regarding the quality of teachers' collective noticing of children's mathematical thinking, *what is the quality of teachers' collective noticing when discussing student work together?* My goal was to characterize teachers' collective noticing, and the extent to which it was grounded in the details of one, sometimes two, student's thinking, as supported by the evidence provided in teachers' conversations. Episodes were holistically coded by considering what details of a student's strategy were discussed, and how teachers' discussion connected to evidence from the student's written work and I considered if and how the groups' discussions (a) described the details of the student's strategy, (b) made claims about how the student solved the problem and what the student understood based on the details of the strategy, and (c) considered how they might respond to the child. However, my holistic coding emphasized how grounded in the details of students' work the teachers' conversations were.

Table 3

Quality of Collective Noticing

	Total Episodes	Duration Range	Average Duration
Robust	71	00:11.8-05:29	01:26.8
Limited	101	00:08.6-05:19.8	01:01
Lack	48	00:05.6-03:36.1	00:47

Three-fourths of the episodes demonstrated some engagement with the details of the student's strategy and less than one-fifth of the episodes contained language that discussed what the child did not do. This suggests that, by and large, even without the presence of a facilitator, teachers were engaged in collective noticing of children's mathematical thinking at some level.

When engaging in discussions about students' written work, teachers were asked to not only attend to the mathematical details of the student's strategy, but also consider potential understandings the child might have based on these details. This attention to understandings helps teachers make instructional decisions that support and extend students' understanding, rather than considering what the child does not understand. This framing takes time to build, as teachers typically discuss students through a deficit lens (Horn, 2007; Louie, 2016), focusing on what students does not understand.

Teachers discussed potential interpretations in approximately one-eighth of the episodes. While teachers were able to discuss the details of the student's strategy in a majority of the episodes, there were relatively few instances when teachers generalized what the strategy details might indicate about the student's understanding. However, this finding is not necessarily surprising considering Jacob and colleagues (2010) found in their STEP study that teachers who had participated in at least four years of professional development on children's mathematical thinking were more proficient at interpreting children's understandings using details from the children's strategies when responding to a written prompt. Therefore, as most teachers had only participated in at most their third year of professional development, it would be expected for teachers to describe the details of a student's strategy without making a connection to what the student might understand based on the detail

Robust evidence of collective noticing children's mathematical thinking. I identified approximately one third of the episodes as demonstrating robust evidence of collectively noticing children's mathematical thinking. When teachers provided robust evidence of collective noticing, the descriptions included mathematically important details such as how children represented the problem context using pictures or numerical representation, and then used these details to describe how the student likely solved the problem. Teachers' discussions in

this category lasted 1 minute and 26 seconds on average (see Table 3), and the majority of the identified details represented a description of the student's strategy rather than the final answer (see Table 4).

Table 4

Focus of Collective Noticing within Quality

	Number of Identified Details	Strategy Details		Answer Details	
Robust	454	402	88.5%	52	11.5%
Limited	355	296	83.3%	59	16.7%
Lack	86	36	41.8%	50	58.2%

For example, the following episode between three teachers, Melissa, Jill, and Shelby (Transcript 2) provides robust evidence of collective noticing. In this episode, Melissa shared how Brayden, a fourth-grade student, solved an equal sharing problem involving four kids sharing 11 pancakes (Figure 11). In my own analysis of Brayden's work, I noticed he used a direct modeling strategy and represented the four sharers as circles and the eleven pancakes as dots within a circle marked "pancakes." Brayden distributed the items, most likely one at a time, until he had three dots remaining. Brayden then redrew the three dots as larger circles, represented as a line around three dots and a line around three circles with a line connecting the two representations. Brayden then partitioned the three remaining pancakes, two into fourths and one into eighths, to share equally with the four children.

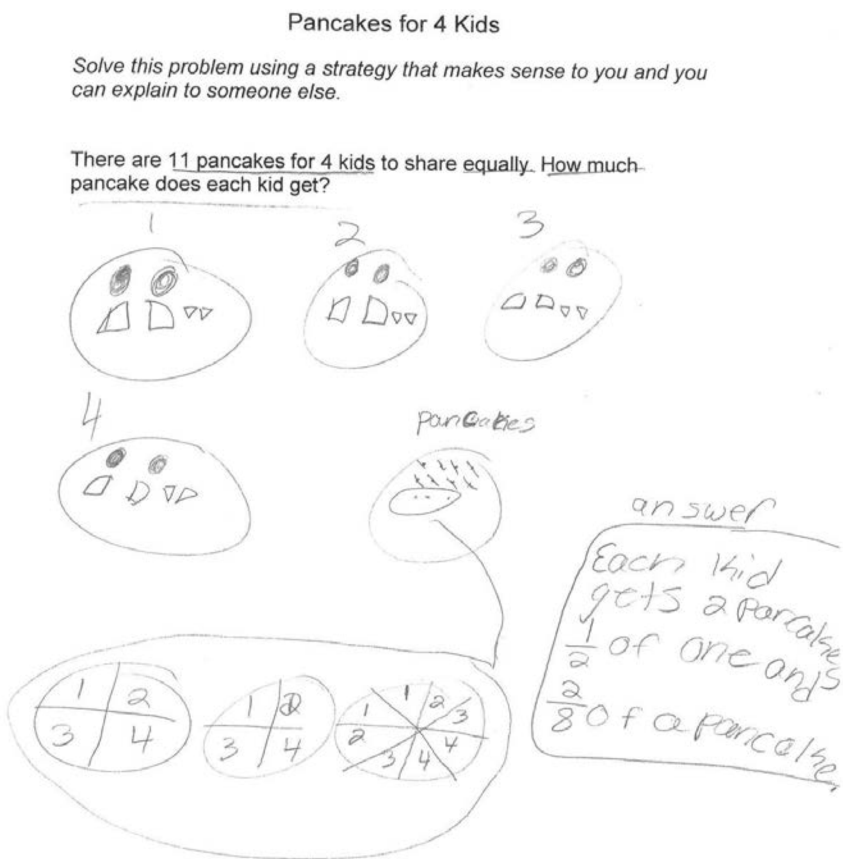

Figure 11. Image of Brayden’s Strategy

Transcript 2: Brayden’s Strategy

1. MELISSA: Brayden started out, and I was surprised by this because he’s a pretty, he’s pretty advanced too and his thinking was different on this. He started here and he drew the circle and put 11 dots and they were the pancakes. And I kind of
2. SHELBY: He just wanted to see ‘em.
3. JILL: Yeah, pretty much all my kids had to draw out the kids and the pancakes. That was one thing I noticed being… You know, they had to see

everything.

4. MELISSA: And then one two three four kids. And so then he put a dot here and he'd mark one out. Put a dot here and mark one out. Put a dot here mark one out. Put a dot here mark one out. Dot dot dot dot.

5. SHELBY: Representing I passed out one pancake

6. MELISSA: And then he got to the three and he came down here and he drew those three pancakes

7. JILL: Oh yeah.

8. MELISSA: And he started with fourths and so each one got a fourth a fourth a fourth a fourth. A fourth a fourth a fourth a fourth. Then he split this one into <<laughs>> eighths. <<laughs>> One-eighth one-eighth one-eighth and so on.

9. JILL: Why did-?

10. MELISSA: So, his answer is two pancakes, a half of one, and two-eighths of a pancake.

11. JILL: Okay.

12. MELISSA: So, I wrote that out on the board two plus one-half plus two-eighths.

13. JILL: Interesting. I wonder why he did that.

14. MELISSA: I know.

15. JILL: Why do you think he did that?

16. MELISSA: I don't know. <<laughs>>

17. SHELBY: Instead of just another fourth?

18. JILL: Yeah

19. SHELBY: Yeah

20. MELISSA: Hmm, interesting

21. MELISSA: I don't know.

22. JILL: Okay.

As seen in Transcript 2, the teachers discussed all of the mathematical details in

Brayden’s strategy, stating how Brayden represented the context and solved the problem. Melissa stated that the dots in Brayden’s strategy represented pancakes (Turn 1) and the four circles represented the kids (Turn 4). Melissa then shared how Brayden most likely marked one dot at a time in to each of the circles representing the kids, marking out each dot as it was distributed (Turn 4), and Shelby added that this process represented Brayden passing out each pancake (Turn 5). Melissa then moved to discuss the three large circles at the bottom of the strategy representing the three remaining pancakes (Turn 6). Her description suggested Brayden most likely partitioned the ninth pancake into fourths and distributed each fourth and repeated this step for the tenth pancake (Turn 8). Melissa then noted Brayden partitioned the 11th pancake into eighths and distributed two-eighths to each kid for a total of two wholes, two fourths, and two eighths (Turn 8). Partitioning this last circle into eighths might be considered atypical, as the child had already partitioned the ninth and tenth pancake by the number of sharers, and this is recognized by Jill and Shelby asking why he partitioned the pancake into eighths rather than an additional fourth (Turns 9-22).

During this interaction, Melissa had an opportunity to verbalize the details of Brayden’s strategy to Jill and Shelby. Melissa connected the details of Brayden’s strategy to the process Brayden most likely used, sharing what the shapes represented within the context of the story and in connection to the numerical quantities. When describing the strategy in this way, Melissa had an opportunity to engage with Brayden’s problem-solving process, or to think as one of her own students. The group’s opportunity to notice is further demonstrated when a strategy detail that might not be considered typical was described. While Melissa described how Brayden partitioned the last pancake, she laughed (Turn 8), which could indicate she thought it was unusual; however it was Jill who made this part of Brayden's reasoning more explicit through her questioning. To notice a detail like this suggests the teachers engaged with a student's informal strategy and recognized that students typically partition wholes while considering the number of sharers, making a connection between the researched-based frameworks for how children solve problems and how their own students solved the problem.

A student's written strategy is a representation of the student's thinking; however, a representation is not always clear or there may be more than one way to interpret how the student solved the problem. Therefore, considering questions that could elicit student understanding is an essential component to noticing children's mathematical thinking and another way teachers engaged in discussions that were grounded in the details of student strategies. Teachers discussed possible questions to pose to a student that referred to a goal of understanding how the student was thinking. When teachers asked questions with this goal, they would often say, "I'm curious to know what her thinking was, because. Did … how did she know it was a fourth?" or "I don't know if she would know that that's the same as 1 1/4...So it would be an interesting question to ask her. Since she put 4/4 + 1/4, if she knows that 4/4 was the whole," or "Then maybe you ask him what's that remainder four? What is that four? And ask him, what does that represent?" These questions are important to pose within this context because it demonstrates how the teachers are engaged in making sense of the student's thinking. Teachers had opportunities to generate questions outside of the classroom context that could be used to elicit students' understandings, and determine next instructional steps, contributing to the teachers making connections to the details of the students' strategies.

Limited evidence of collective noticing of children's mathematical thinking. While a third of the examples showed evidence of robust noticing, I identified roughly one-half of the episodes as showing limited evidence of noticing children's mathematical thinking. When teachers provided limited evidence of discussing children's mathematical thinking, the descriptions included some of the mathematically important details, but in ways that were isolated from the student's problem-solving process. That is, the details were described in a way that made it difficult to reconstruct the student's strategy. On average, teachers sustained discussions that were almost 30 seconds shorter than the discussions that provided robust evidence of discussing children's mathematical thinking, or about 1 minute in length (see Table 3). Similar to discussions providing robust evidence, teachers did mostly engage with the strategy details; however, their discussions included fewer overall strategy details over a larger number of episodes and their focus shifted slightly more toward discussing the students' final answers (see Table 4). Therefore, while episodes in this category included teachers' capacity to notice children's mathematical thinking and make this noticing visible to their group, there were still opportunities to make this noticing more explicit and make connections between the details of the strategy to the student's problem-solving process.

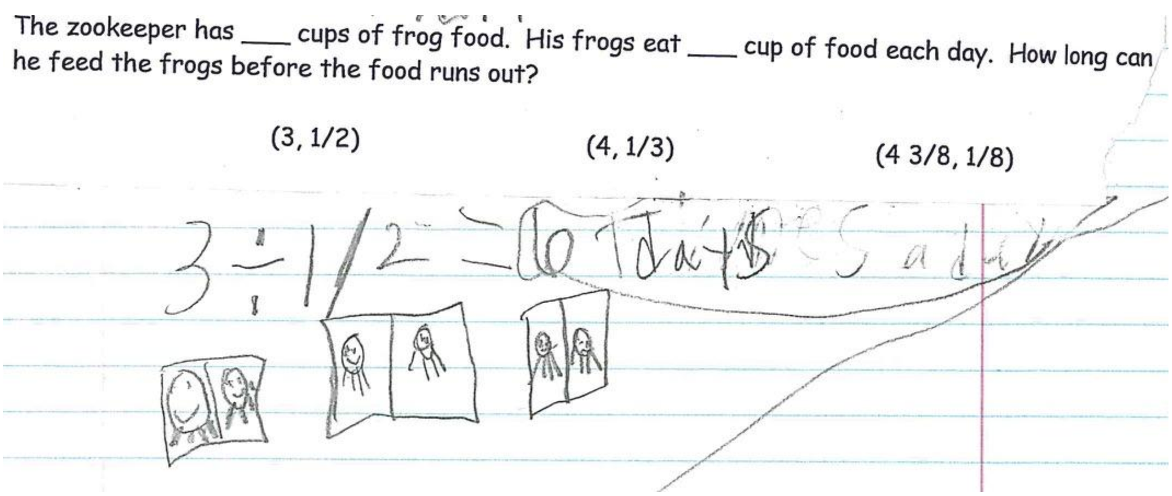

Figure 12. Image of Ethan's Strategy

Transcript 3: Ethan's Strategy

1. JILL: I thought this one was interesting because, look how he notated that 3 divided by half equals 6 days.
2. MELISSA: I had a couple do that too.
3. JILL: And he struggled with it. I don't know…he just, he understands you know that he divided three cups into two parts, into half. And he was able to count that up.
4. MELISSA: That's good
5. JILL: But, yeah.
6. MELISSA: Little people are cute too.
7. JILL: <<laughs>> That's Ethan.

As an example of limited evidence of collective noticing, consider another complete episode from Melissa and Jill. In this episode, Jill described how her student Ethan, a third grader, solved a multiple-groups measurement division problem, solving for the number of ½ cup servings in 3 cups (3 ÷ 1/2, Figure 12). Ethan most likely solved the problem initially using a direct modeling strategy, drawing three rectangles to represent three cups of food, partitioning each rectangle to show a half-cup of food, and then counted the number of halves in three. Ethan also represented this problem using a division equation.

In this episode (Transcript 3), the teachers mentioned most of the mathematically important details in Ethan's strategy. Jill began the interaction by pointing to the division equation $3 \div 1/2 = 6$ Ethan notated (Turns 1–3), and Jill was most likely pointing to the model when she states that Ethan understood "that he divided three cups in two parts...And he was able to count that up" (Turns 5–9). However, recall one of the purposes of asking teachers to participate in these discussions was to help teachers verbalize what they noticed about the child's mathematical thinking. And while Jill may have noticed this student used a direct modeling strategy, she did not mention that Ethan represented the three cups as rectangles. Jill's claim in Turn 3 that Ethan understood how to divide the cups in half is vague and could be interpreted a number of ways. Jill's statement could be interpreted as (a) Ethan made two groups of 1 1/2, or (b) Ethan mentally divided three by one-half. This is an important distinction, because the type of strategies children use typically indicate a level of understanding and inform the types of problems teachers might ask students to engage with next. In this episode, it is possible Jill could over-generalize Ethan's understanding and pose a question outside of his understanding, although a teacher would need to question Ethan and provide more problems to better understand how he is thinking about the relationship between a half and a whole and if he no longer needs a picture representation to show his thinking.

Within the data, there existed a subset of episodes within this category in which most of the mathematical details were described, and the strategy could generally be recreated without looking at the student's written work. However, I considered this subset of sessions as providing limited evidence of collective noticing because there were key descriptions within the student's strategy that teachers did not mention. These key descriptions failed to connect the student's representation to the mathematical processes the student used to solve the problem.

As an example of the details as isolated from the process, consider an episode from Melissa and Jill with a different problem type (Transcript 4). In this episode Melissa described how Kelly, a fourth-grade student, solved a multiple-groups multiplication problem, five aquariums with 3 1/2 gallons of water each (Figure 13). Kelly represented the problem as a multiplication equation. She did not need to model the problem using pictures, but rather used symbolic notation decomposing each 3 1/2 into a 3 and a 1/2. Kelly then used a repeated addition strategy, adding five groups of 3 and five groups of 1/2 and then combining the subtotals for a final answer of 17 1/2.

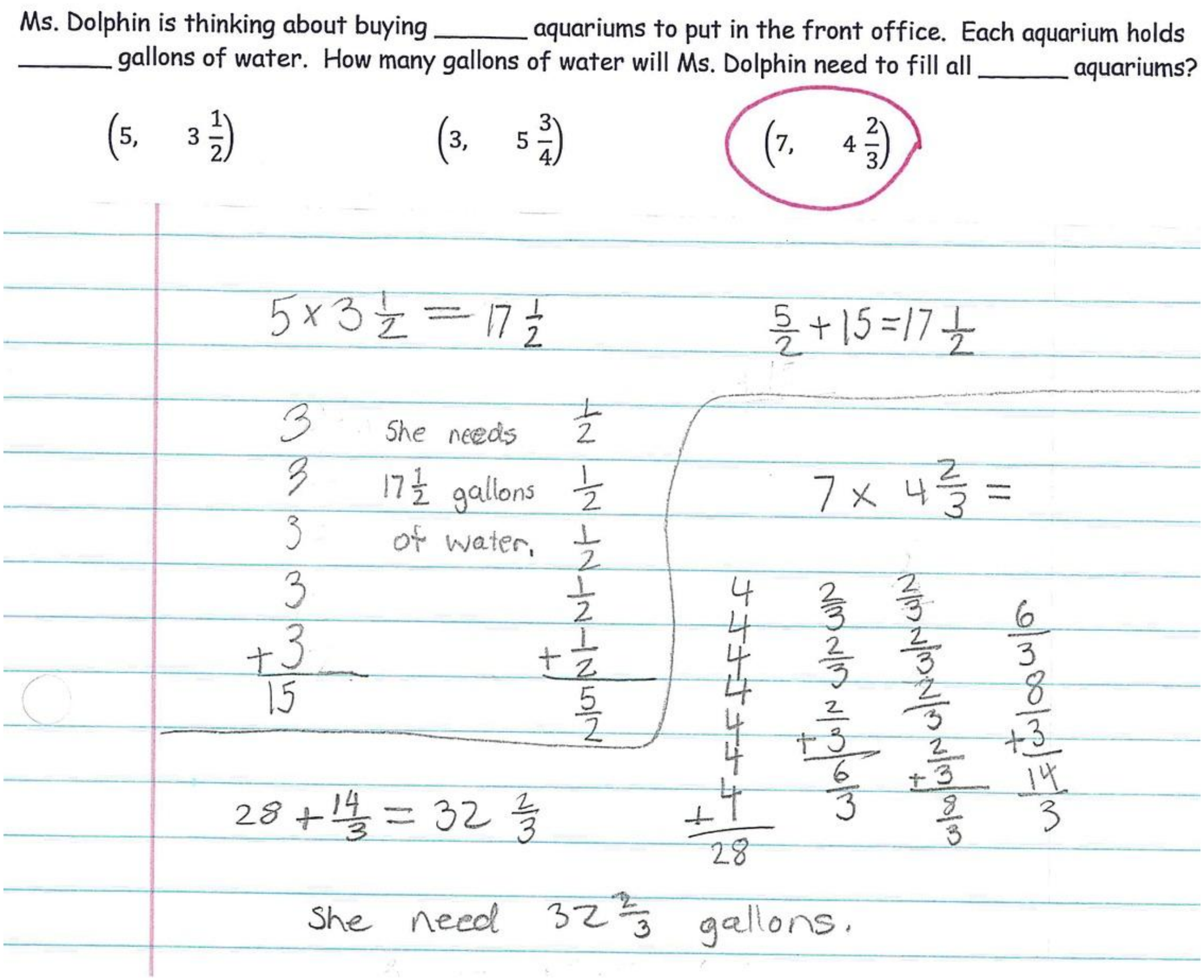

Figure 13. Image of Kelly's Strategy

Transcript 4: Kelly's Strategy

1. MELISSA: I had quite a few do this, the five times. I'm surprised she wrote it. I'm surprised when they write it like that. Cause we haven't done a whole lot of writing it like multiplication problems.
2. JILL: I had one do that too. Well, actually she did-.

3. MELISSA: But then she did is she went back and she did the five-, added three fives times and then added a half five times.

4. JILL: She got fifteen and five halves.

5. MELISSA: She did 5-half plus fifteen equals 17 and a half. So. And she did the same thing when I gave her-

6. <<Jill begins to discuss a strategy from her class>>

While Jill and Melissa discussed important mathematical details and the strategy could mostly be recreated, neither explicitly mentioned some key details in relation to Kelly's thinking. For example, while Melissa noted that Kelly added the group of whole numbers separately from the group of fractions (Turn 5), Melissa did not articulate the detail that Kelly decomposed 3 1/2. In addition, neither Melissa nor Jill mentioned that Kelly was able to add an improper fraction, 5/2, to a whole number, 15, mentally coordinating five halves as two wholes and one half. While these details may seem negligible, recognizing these details can provide a teacher with evidence for some potential understandings Kelly has, such as understanding mixed numbers and improper fractions as numerical quantities that can be represented in a number of ways (5/2 = 4/2 + 1/2 = 2 + 1/2 = 2 1/2).

It is important to note that I do not always expect teachers to discuss every student's mathematical thinking in robust ways. Teachers were asked to choose strategies from their classroom that represented a variation in student thinking. Therefore, teachers may have implicitly or explicitly connected details across strategies and may have chosen to highlight a particular aspect of a student's strategy. For example, Jill and Melissa discussed Ethan's strategy

(Transcript 3) in a way that I identified as limited; however, prior to this episode they had discussed two similar strategies in a robust way. When Jill began the discussion of this episode, she noted that the student had used a division equation, focusing on a piece of the strategy that was different from the other two. Therefore, while it may be worthwhile to consider how episodes related to one another, this study was designed to consider episodes as isolated units of analysis unless teachers called attention to another student's strategy.

Lack of evidence of collective noticing of children's mathematical thinking. I identified approximately one-fifth of the episodes as demonstrating a lack of evidence of collective noticing of children's mathematical thinking. When teachers discussed student strategies in ways that did not include a focus on the details (see Table 3), their descriptions on average included two details related to the student's strategy but usually in vague ways, and one detail related to the student's final answer. In addition, almost one-fourth of the episodes in this category included teachers making claims about student understanding or deciding how to respond without connecting their claims or decisions to the details of students' thinking. While these episodes were on average about the same length as episodes coded as limited, teachers may have shifted their discussions to engage in other ideas outside of noticing children's mathematical thinking, before returning to the student.

For example, consider the following piece of student work Melissa shared with her group (Figure 14). Kendall, a fourth-grade student, solved an equal sharing problem, 6 share 8, using a direct modeling strategy, sharing groups of items. Kendall represented the six monkeys as rectangles and the eight bananas as lines. Her strategy is organized in a way that could help her use proportional reasoning, as the monkeys are represented as two groups of three and the bananas are represented as two groups of four. Kendall then mentally partitioned the bananas, or lines, into thirds, demonstrating the distribution of thirds as dots and sharing two bananas among the six monkeys. Kendall then wrote a final answer of 4/3 or 1 1 /3 bananas for each monkey.

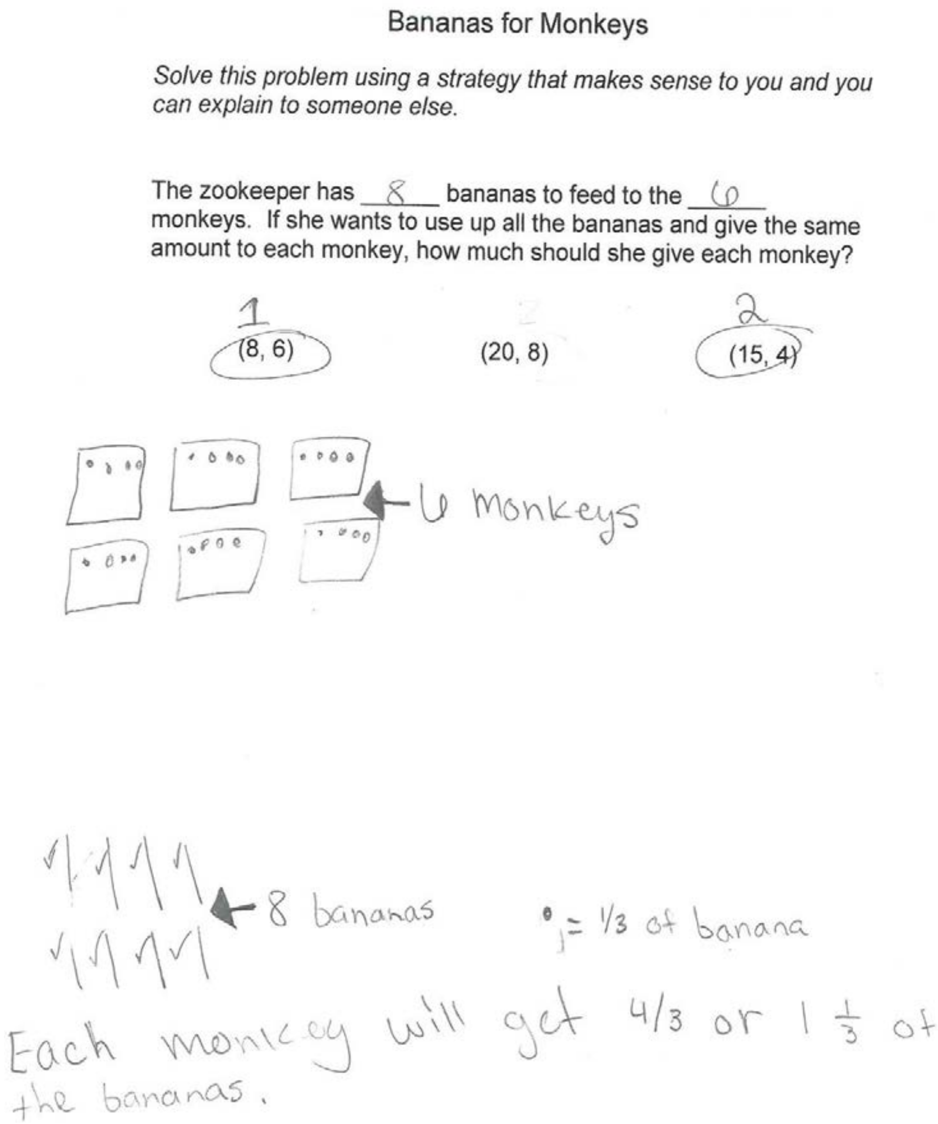

Figure 14. Image of Kendall's Strategy

Transcript 5: Kendall's Strategy

1. MELISSA: This one was each would get 4-thirds or 1 and 1-third, I thought that was pretty strong strategy.

In this episode, Melissa only attended to the answer detail of Kendall's strategy (see

Transcript 5), repeating that Kendall answered with two equivalent quantities and stating she believed this to be a strong strategy. However, Melissa did not verbalize any of the details Kendall used to solve the problem or why she believed this to be a strong strategy. While Shelby and Jill might assume what Melissa meant, the teachers did not discuss how Kendall's direct modeling strategy could indicate a different understanding then other students who used more typical direct modeling strategies, because Kendall mentally partitioned the *bananas*, rather than modeling the partitions. As Melissa did not mention these details, only focusing on Kendall's answer, this episode was categorized as a lack of evidence of collective noticing.

Additional discussion tendencies. In some episodes across the three categories of quality of collective noticing, teachers discussed students' work in ways that were not grounded in the details of the written strategies. I identified four themes, which included discussions of (a) what the student did not understand, (b) claims about student understanding that were not supported by strategy details, (c) how the teacher takes over the child's thinking, and (d) broad instructional implications. In addition, there were also a few episodes where the teachers did begin with a discussion of student work, but the conversation shifted to another purpose, such as instructional implications or understanding mathematical content, before returning to the discussion of student work. However, while these instances were documented, these tendencies were a minority within the episodes as most episodes were grounded in the details of the student strategy.

Focusing on what the child does not understand. The professional development in which teachers were participating was designed with a strong commitment to focusing on strengths in students' thinking. Nonetheless, approximately one-tenth of the episodes contained discussions about what the student did not understand, suggesting a possible orientation to thinking of students in terms of deficits. In these episodes, teachers made generalizations about students' strategies such as "There is just a problem with her depth of understanding," "He just doesn't have an understanding of numerators and denominators and what they even mean," and "He didn't really have any anticipation." While these generalizations may have some merit, they make it difficult to decide what an appropriate next step might be if what the student does knows has not been articulated in order to build from there.

Broad interpretations of student understanding. In addition to considering what the student did not understand, teachers also made general interpretations about their students' understanding; however, it was not always clear what they believed the student understood or how these interpretations connected to the details of the student's strategy. When teachers discussed what students understood in this way, they made general statements: for example, "she has understanding of fractions," or "he has an understanding." While these broad interpretations indicate the teachers are considering what their students may understand, they were not explicitly connected to the details of the student's strategy. In order to build on student understandings, interpretations should be grounded in the children's thinking, as evidenced by either the details of the child's strategy or questions that elicit children's thinking.

Directive actions. Another way teachers did not provide evidence for engaging in children's thinking is when teachers talked about ways they would use questioning to guide students to a correct answer. Teachers described questions such as "'How many sandwiches would we need to have for each child to get a whole sandwich?' Hopefully he would say eight. 'Do we have eight? No, so can they get a whole sandwich?'" or "Barry needs to go back and understand what the story is saying. Just go back into and actually have it modeled. What is the story saying? Where are your three friends? Where are your four granola bars? Now how can you share them?"

When students struggle, teachers are encouraged to elicit student understanding and build on, or support, that understanding. Teachers can do this by asking questions that do not attempt to take over student thinking and encouraging the student to solve the problem in a way that makes sense to the student rather than the teacher. During the professional development, teachers were encouraged to support students by first ensuring that the student understood the context of the problem in a way that allowed the student a way to enter into the problem. Teachers were then encouraged to ask students to connect details of their strategy to the problem context. In the examples provided, teachers were suggesting questions that encouraged the student to consider the context of the problem; however, the questions were framed more as a directive, almost as an attempt to take over student thinking rather than elicit and build on to student thinking. Therefore, asking these types of questions provides opportunities to notice children's mathematical thinking during classroom instruction.

Instructional implications. Another way teachers' discussions focused on ideas outside of noticing children's mathematical thinking is when teachers used a student strategy to represent a group of students and engaged in a broader discussion about student understandings and instructional implications. For example, Janice and Molly used Damien's strategy (Figure 15) for a multiple-groups measurement division problem, the number of half- cup servings in three cups, to discuss how children conceptualize, as either a quantity or a verb. Janice stated:

I don't know if they're thinking about it as fractions? There is something there, there is something there that when the say half, I think because it is a word rather than a number because you cut your sandwich in half, you know? So I think there is a disconnect on what a half is.

Janice then shared an interaction she had with student in class to help him solve the problem successfully. Janice shared that she notated the 1/2 to connect the action of partitioning the rectangles to the quantity and numerical representation of 1/2. The episode ended when Janice wondered if changing the story problem to read as *His frogs eat a half cup of food each day*, instead of *1/2 cup of food*, would have made the problem easier for students like Damien to conceptualize. Therefore, while this episode was coded as a lack of discussing Damien's mathematical thinking because this was a strategy Janice created with Damien, the conversation between Janice and Molly are important for considering how teachers use student thinking to make instructional decisions.

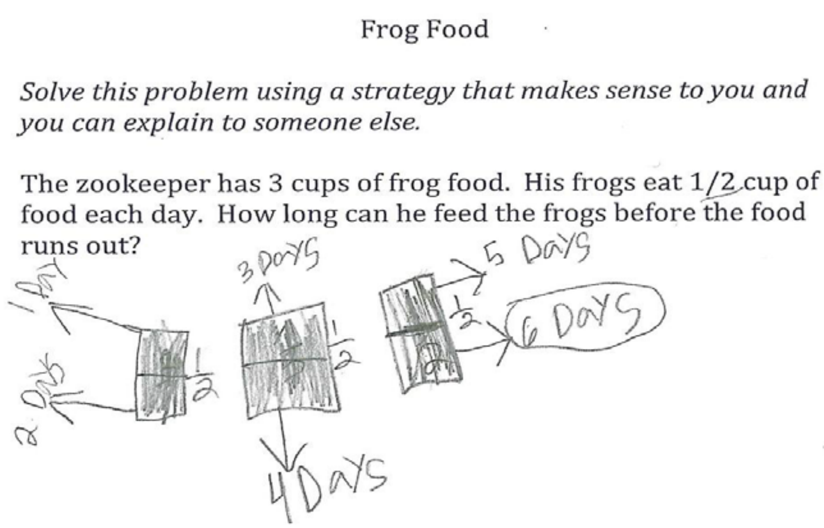

Figure 15. Image of Damien's Strategy

Summary of the quality of collective noticing. In Figure 16, each of the 29 sessions that I analyzed is represented along the horizontal axis. Within each session, the number of episodes that were coded as providing robust, limited, or lack of evidence of collective noticing are shown with blue, green, or yellow bars, respectively. There are two notable patterns. Twenty-six of the 29 sessions included at least one episode characterized as robust evidence for understanding and seven sessions contained more episodes demonstrating robust evidence of collective noticing than limited or lack. These findings suggest most of the teacher groups had productive discussions within the sessions and demonstrated the capacity to participate in discussions that were grounded in the details of student strategies without the presence of a facilitator. Regardless of the year in professional development, teachers were providing robust evidence for discussing children's mathematical thinking in their first year of the collaborative inquiry sessions. In the next section, I will present

the patterns of interactions conversations that demonstrated how teachers engaged in collective noticing.

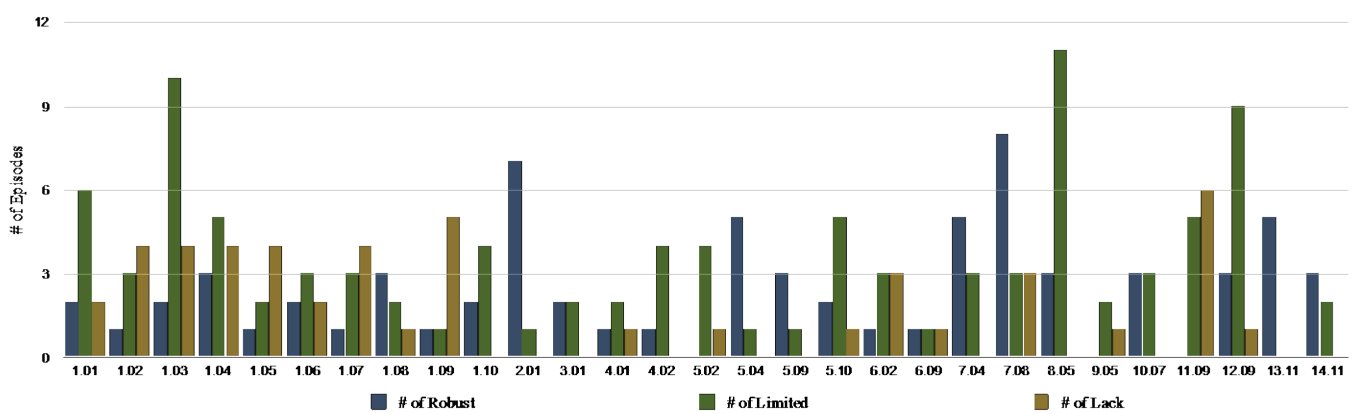

Figure 16. Quality of Collective Noticing by Session

Patterns of interaction in collective noticing

In this section, I investigate the second sub-question of my research question, *what are the teachers' patterns of interaction in collective noticing when discussing student work together?* I began by reviewing the same episodes to explore how teachers interacted while engaged in the collective noticing of individual student's mathematical thinking. I read through the transcripts to identify exchanges within each episode that demonstrated how teachers interacted with one another to make sense of a student's strategy together and then looked across all episodes to identify variations in how teachers engaged with one another when noticing children's mathematical thinking.

The teachers' interactions took on a range of forms that I categorized as either creating an opportunity for teachers to either collectively notice children's mathematical thinking by jointly constructing the students' strategies, or an opportunity for one teacher to independently construct students' strategy by verbalizing what he or she noticed about a child's mathematical thinking.

In the first category of interactions, when teachers jointly constructed collective noticing, most often as the student's teacher (or the speaking teacher) described or made claims about the strategy, the partner teacher provided elaborations or additional details that supported the claims or continued the idea. There were also a number of instances where the partner teacher elaborated on what the speaking teacher noticed by stating either a claim or counterclaim. Occasionally the partner teacher would offer an idea about how they might respond to the student, in response to either a description or claim the speaking teacher made, or would make a connection to another student that was previously discussed or was similar to an idea observed in his or her own classroom.

In the second category of interactions, one teacher had an opportunity to verbalize what they noticed about a child's mathematical thinking; that is, the speaking teacher independently constructed the student's strategy while the partner teacher listened. During these interactions I identified a number of instances when the partner teacher would jump in to either complete or repeat a description or state agreement, most likely as an indication of listening attentively. I also identified a number of interactions that allowed the speaking teacher to either reflect on or refine his or her noticing when asked to clarify a detail.

Patterns of interaction: Joint construction. About one-fifth of the episodes included at least two instances of high contributions, which could open up an opportunity for the group to co-construct noticing children's mathematical thinking. Instances of high contributions generally included the ways teachers either reflected or expanded upon one another's ideas. When teachers engaged with one another in this manner, they had an opportunity to jointly construct their noticing of the mathematical thinking of their students. Reviewing the episodes of discussing children's written work, I identified contributions (see Figure 10) related to the teachers noticing children's mathematical thinking that demonstrated a higher contribution quality and could contribute to the joint construction of noticing. In this section, I present examples of how both teachers contributed to the collective noticing of children's mathematical thinking by elaborating on the details and offering claims and counterclaims that interpret the details of the child's strategy.

Elaborations: Re-creating the student's strategy together through elaborations. Most of the instances of high contributions I characterized as the speaking teacher either describing the details of or making claims about the student's strategy, and the partner teacher contributing to the discussion by furthering the claim or detail. When teachers engaged in this way, they had an opportunity to build on what their partner teacher was noticing, allowing them to make the details more explicit.

Consider an example of how teachers recreated a student's strategy together. Debra and Claudia discussed Avery's third strategy for the equal sharing problem, 8 share 5 2/4 (Figure 17). Avery used a repeated halving direct modeling strategy that did not solve the problem and provided an answer of 3/4. Avery drew six squares and partitioned each square into fourths.

With the sixth square, Avery marked out two-fourths of the square in order to represent the peanut butter sandwiches. She then distributed each fourth one at a time to each of the eight children. She continued to distribute fourths until the last half square remained. Because four children received three-fourths and four children received two-fourths, Avery partitioned the remaining half into fourths (or eighths of a whole) and then distributed each of the shares to the remaining four children. Finally, Avery counted that each child received three pieces, or 3/4.

8 children want to equally share 5 2/4 peanut butter sandwiches with no leftovers. How much can each child have?

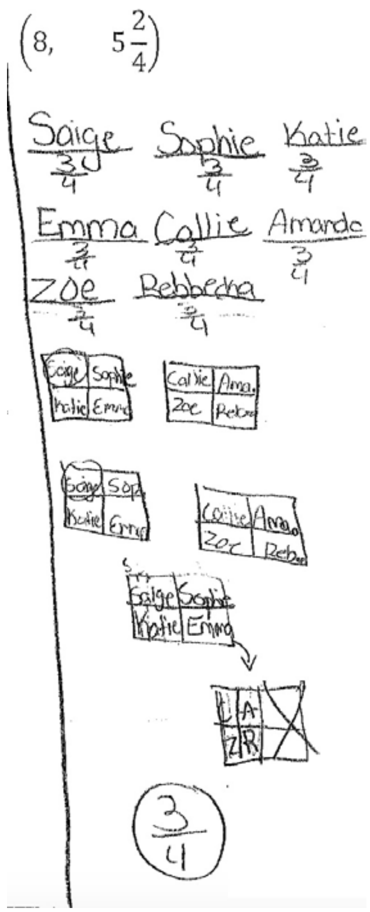

Figure 17. Image of Avery's Strategy

Transcript 6: Avery's Strategy

1. DEBRA: Then, for the last one, that was just way too difficult for her. She was a little out of her league with that one. She did draw the five sandwiches, she did have eight kids named, and she did divide them into fourths, and gave—
2. CLAUDIA: I wonder if she almost sees like that Dillon, the one that I saw, cuz he saw three-fourths, too.
3. DEBRA: Yeah. Oh, see, she divided them all into fourths. Then, there's eight kids, so those are eight kids there, eight kids there, and then she has one, two, three, four, five, six, seven, eight kids there.
4. CLAUDIA: She's given smaller pieces to four of the kids.
5. DEBRA: Yeah, and she even, this time, has it drawn that yeah, that is a half of a sandwich, or two-fourths. She cut it into eighths, but on one side, but not on the other side. It's just going to be a matter of showing her, oh, wait a minute, is that fair? Then I think she'll have it.

Debra began the interaction stating that the problem was too difficult for Avery, and Claudia connected Avery's understanding of three-fourths to a previous episode (Turn 2) in the session (see Transcript 6). Debra seemingly acknowledged this claim but shifted back to Avery's strategy in Turn 3 by attending to the detail of the number of fourths that Avery distributed. As Debra began to count the last four pieces, Claudia elaborated on this description in Turn 4, stating that the last group of pieces Avery distributed was smaller than the other fourths. In Turn 5, Debra agreed with this description and explicitly named the fractional amount of the last four shares as fourths. This interaction allowed both teachers an opportunity to articulate the details of the student's strategy, or make sense of the student's strategy together.

When teachers made sense of the student's strategy together, each teacher took an opportunity to build upon what was noticed through the addition of details that the other teacher may not have considered or had not explicitly stated. More than half of the episodes coded as robust had at least one interaction where a partner teacher elaborated on a strategy detail, suggesting that these types of interactions could open up opportunities for teachers to make their noticing of children's mathematical thinking visible to one another.

Descriptions and claims: Invitations to participate in collective noticing. Occasionally as teachers began to discuss a student's strategy, they expressed some confusion or curiosity about a detail. While these instances may not have always been posed as a question to the partner teacher, these interactions created an opportunity for the partner teacher to state what he or she noticed about the student's mathematical thinking. Teachers may have stated, "I'm not sure where [a strategy detail] came from," or "I can't figure out why [the student]." These questions often encouraged the partner teacher to contribute to the collective noticing of the student's mathematical thinking, providing their own claims about how the student may have solved the problem based on the details, and could lead to an interaction where each teacher begins to elaborate on the details.

Claims and counterclaims: Offering alternative interpretations. In addition to contributions that allowed teachers to jointly construct collective noticing by both teachers through elaborations, I also identified turns where teachers offered alternative claims or perspectives. Alternative perspectives were important for collective noticing because they offered an opportunity to return to the details of the strategy to support claims, or consider more than one explanation for the student's strategy. For example, consider an excerpt from Claudia and Debra discussing Dillon's strategy for the equal sharing problem, 8 share 5 2/4 (Figure 18).

8 children want to equally share 5 2/4 peanut butter sandwiches with no leftovers. How much can each child have?

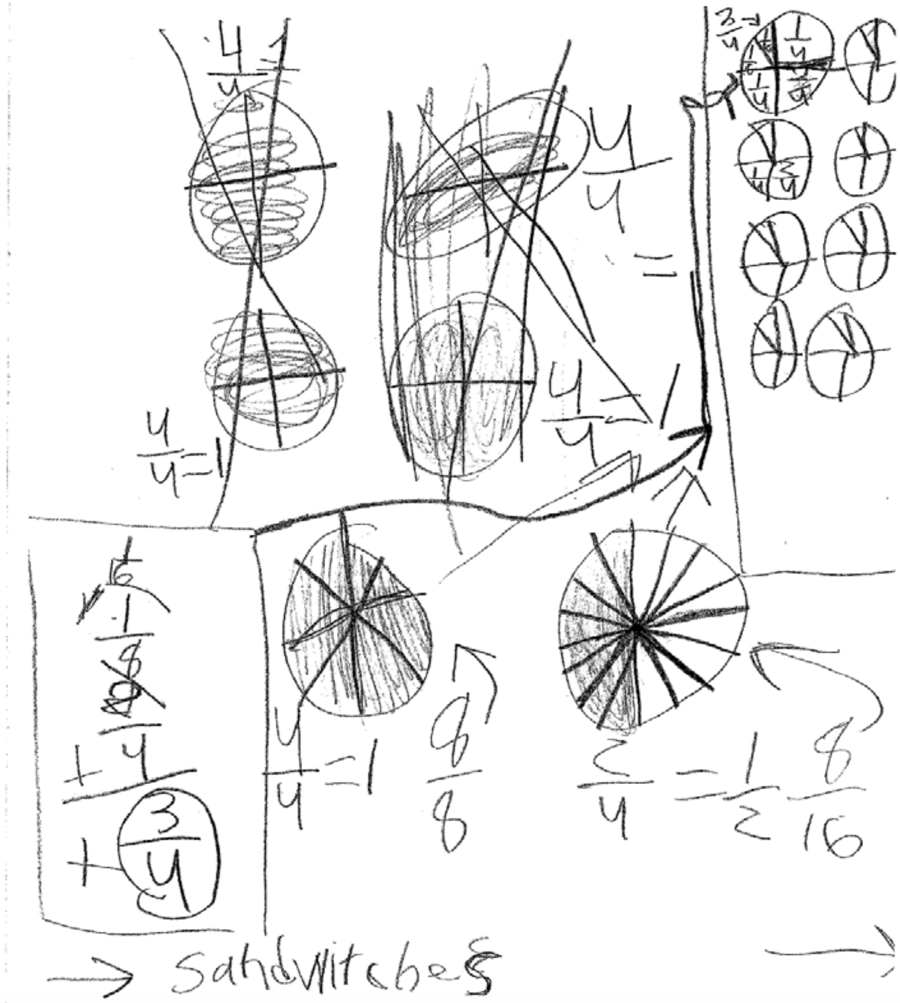

Figure 18. Image of Dillon's Strategy

Transcript 7: Dillon's Strategy

1. CLAUDIA: I was trying to steer him towards that [sixteenth] would be equivalent to—as far as eighths, what would that be equivalent to? He just was having a hard time with it.

2. DEBRA: Okay, I don’t know fourth-grade standards very well at all. In fourth grade, how much do you guys have to do with fractions? Is it just recognizing them?

3. CLAUDIA: No, we do a significant amount with fractions, especially third, fourth quarter. We’re working a lot on equivalent fractions. They have to recognize equivalent fractions.

4. DEBRA: Because I think he does.

5. CLAUDIA: Well, he does, but when he was putting the pieces together, he wasn’t putting them together correctly. He was not fully understanding. He has it, but even with my guidance he was calling the answer three-fourths. Everybody gets three-fourths.

6. DEBRA: Oh, okay.

7. CLAUDIA: He couldn’t get away from that, no matter how much I tried to talk to him. I’m stumped on how do I help him get there? Where do I go from here? Because he has the idea where they would each get two-fourths, which would be one-half, plus they would each get one-eighth, right? They would each get, I guess 2/16. Wait, is that right? One, two, three. Oh, 1/16 because it was half of a sandwich that was left, so 1/16, but then he had a hard time converting that together.

Dillon’s strategy may look confusing, but he used a direct modeling strategy to solve the problem. In the middle of the page, he drew six circles partitioned into fourths to represent the pancakes. He then distributed the fourths one at a time to each of the eight children until he had distributed two-fourths, indicated by the shaded and marked out circles. There are eight circles on the right side of the page that show the same partitions and two of the strategies show the notation of 1/4 and 2/4. On the fifth circle, Dillon further partitioned the wholes into eighths and distributed 1/8 to each of the children, indicated by the notation of 1/8 in the top circle. Dillon then further partitioned the last whole into sixteenths, so he could distribute each of the eight-sixteenths; this notation is also indicated in the top circle representing the children. Lastly,

Dillon wrote an equation to represent one share, 1/16 + 1/8 + 2/4 = 3/4. The circles indicate Dillon potentially understands the relationship between a unit fraction and a whole, or that four-fourths, eight-eighths, and 16-sixteenths equal a whole. Dillon also may understand unit fractions in relation to a half, as he was able to share a half equally and notate that 2/4 = 1/2 and 2/4 = 8/16.

Claudia began the episode sharing the interaction she had with Dillon as he described his strategy to Debra (Transcript 7). In this excerpt, Debra claimed in Turn 4 that she thought Dillon understood equivalent fractions, possibly because Dillon was able to notate 2/4 = 1/2 and may have considered how these numbers were equivalent to 8/16. While Claudia acknowledged Debra’s claim in Turn 5, she then suggested that if Dillon had a strong understanding of these relationships, he would have been able to combine the different fractional amounts for a final answer of 11/16, or he may have stopped to considered how 1/16 + 1/8 does not equal 1/4. Claudia’s claim allowed the pair an opportunity to consider what Dillon potentially understood about fractions, using evidence from the written strategy.

Claims and counterclaims: Reframing conversations. Another interaction that may open up opportunities for teachers to co-construct noticing is when one teacher states something that a student did not do, and the other teacher elaborates to state what the student did do. Within the robust category, teachers discussed what students did not do in eight of the episodes. However, in one episode, the teachers discussed how the student did not model the problem to suggest he had a higher level of understanding, and in the other seven episodes, another teacher would counter with what a student did do.

For example, consider an interaction between Ronda, April, and Sally as they discuss Gabriel's strategy for an equal share problem, 4 share 11 (Figure 19). Gabriel solved the problem using a direct modeling strategy, representing each of the 11 pancakes as a rectangle and partitioning each rectangle by the number of sharers, or into fourths. Gabriel then numbered each partition 1-4, representing a distribution of one-fourth from each rectangle. Lastly, Gabriel wrote an equation, adding 11 groups of 1/4, for a total of 11/4.

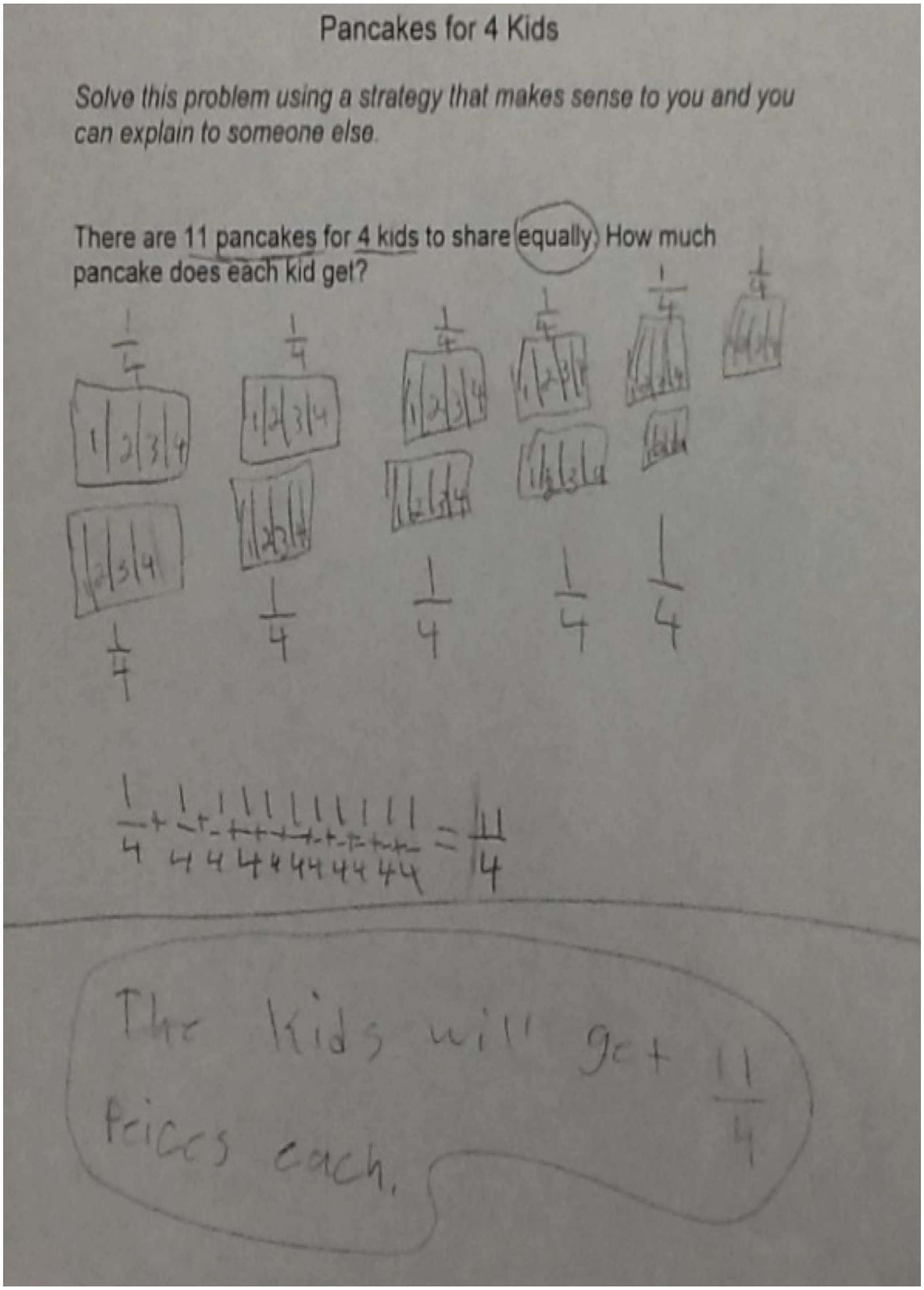

Figure 19. Image of Gabriel's Strategy

Transcript 8: Gabriel's Strategy

1. RONDA: Okay. My kiddo Gabriel, he was pretty much a one-to-one correspondence kiddo. He drew his eleven pancakes. Knew there were four kiddos and so he divided them into fourths. And then he went back and just used his number sentence to get to 11-fourths. But he was not able to tell me that it was two and three-fourths of an actual whole. So.
2. APRIL: But at least he knew to add them.
3. RONDA: Right. And his number sentence does match his picture which is a good thing.
4. APRIL: Yep. And then he also showed that each student-, one person is gonna get a fourth from each.
5. RONDA: Yeah, he just drew it for one kid.

In Turn 1 (Transcript 8), Ronda described Gabriel's strategy, ending with a statement that Gabriel did not demonstrate if he knew that 11/4 was equal to 2 3/4. In Turn 2, April made a conversational move that stated what Gabriel did know, that he needed to add the number of one-fourths. April's conversational move may have helped extend the conversation, encouraging Ronda in Turn 3 to state that Gabriel was able to write a number sentence that matched his strategy, a connection she had not made when she mentioned his equation during her initial description.

While these types of conversational moves did not happen often, they demonstrate the potential that other teachers might have in helping to reframe the conversation from what children did not or could not do, to what children are able to do. This is an important distinction for teachers to make when noticing children's mathematical thinking so that teachers can make instructional decisions that build from children's understanding. For example, because Ronda and April recognized Gabriel was able to decompose a whole into four fourths, Ronda could pose a problem to Gabriel that would encourage him to compose fourths into wholes.

Claims and counterclaims: Negotiating mathematically important details. Another way teachers used the idea of what a student did or did not do was to consider and negotiate what counts as an important mathematical detail. Interactions like these can open up opportunities for teachers to determine which details in the student's strategy are important to attend to in order to interpret potential student understandings.

For example, consider another example from Debra and Claudia. Claudia is sharing Conner's strategy for solving an equal sharing problem, 8 share 5 2/4 (Figure 20). Conner solved the problem using a direct modeling strategy, drawing five circles and then two quarter-circles.

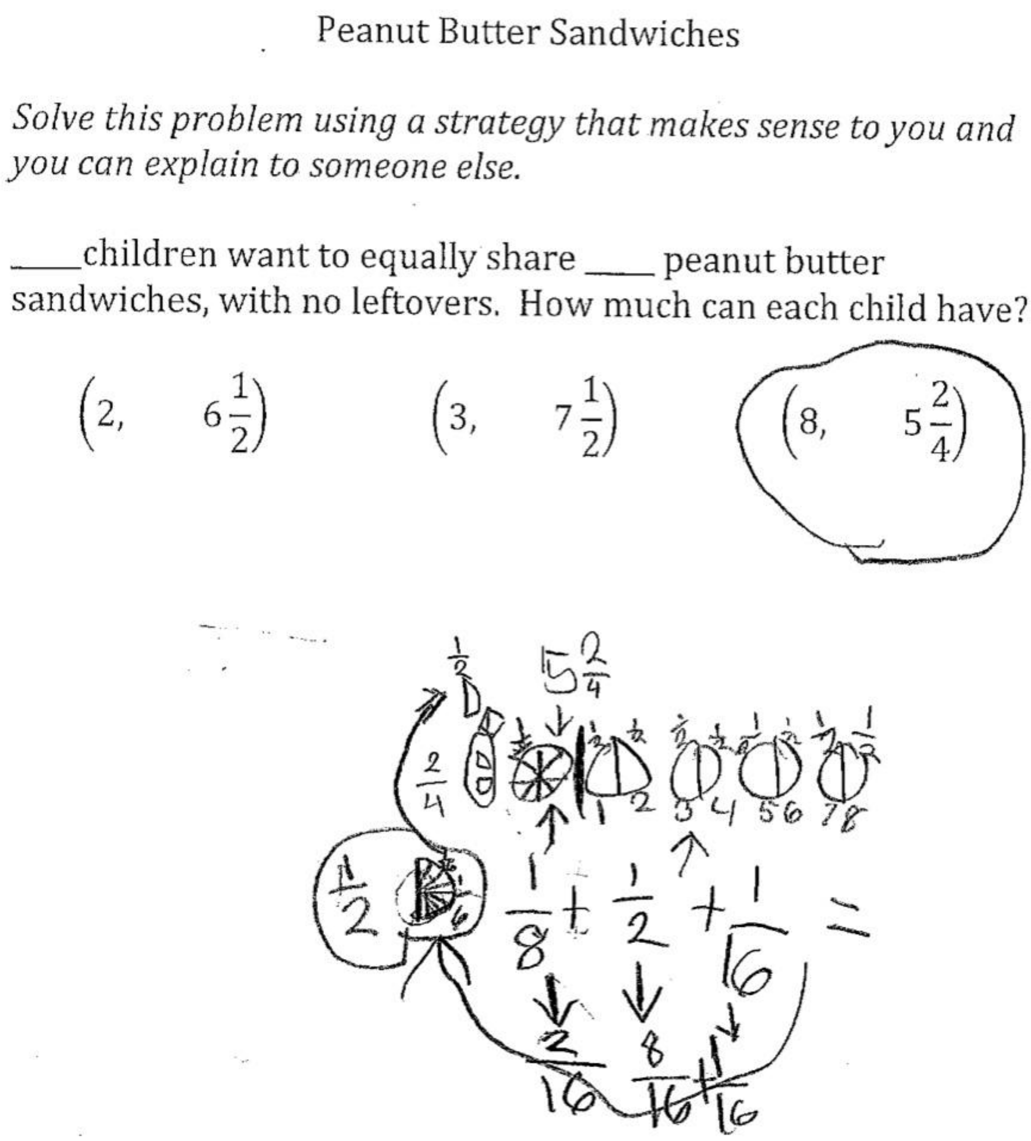

Figure 20. Image of Conner's Strategy

Transcript 9: Conner's Strategy

1. CLAUDIA*:* Conner probably does not need to draw pictures, but I think sometimes he is drawing pictures to show each part. You can see two, three, four, five, and

two-fourths, and he’s drawing little tiny pieces, instead of showing what it is. He was showing breaking it up into the halves, and showing each one gets one-half. One, two, three, four, five, six, seven, eight. Then, from here, he was showing that each student gets one-eighth. From here, I think he was taking those 2/4 and breaking it up and showing that that was 1/16. He put all of that together and converted them into 16ths. What he didn’t do is combine them. His answer is there, he has 11/16, just that little tiny step further.

2. DEBRA*:* He just didn’t—yeah, we’re at the sum.

3. CLAUDIA*:* Right. What is that all together? I think with

4. DEBRA*:* Isn’t that interesting, cuz he doesn’t have the plusses between those. Well, yeah, he does right there.

5. CLAUDIA*:* He has one of the plusses. He had them up here, but was showing that he did convert them to equivalent fractions.

6. DEBRA*:* Huh. Okay.

7. CLAUDIA*:* He didn’t show how, though, so that would also be a question, to say, “How did you know that one eighth was equal to—how did you know?”

8. DEBRA*:* 9. Yeah, I would question it.

10. CLAUDIA*:* Maybe not to show in a picture, but was he multiplying, cuz I think that’s what he probably was doing.

11. DEBRA*:* Yeah, but not letting him get away with that just now.

Conner partitioned four circles into halves, distributing one-half to each of the eight sharers, one circle into eighths, and then redrew the two quarter-circles as a half-circle and then made eight partitions, notating one of the pieces as 1/16. Conner then notated his distribution as 1/8 + 1/2 + 1/16, and also drew arrows with a second set of fraction notations 2/16, 8/16, and 1/16.

In Turn 1 (Transcript 9), Claudia described Conner's strategy, noting at the end that he converted 1/8 and 1/2 into sixteenths, but that he did not combine the fractions for one final answer. In Turn 4, Debra states that Conner did not write addition signs in between the second set of fractions, perhaps suggesting a reason that Conner did not combine the fractional amounts. In Turn 5, Claudia refers to the initial expression with addition signs, suggesting that he was not rewriting the expression but notating the equivalence relationship. With this counterclaim, Claudia may have been suggesting that the lack of addition signs was not a mathematically important detail because Conner's strategy demonstrated understanding of equivalence. Discussing the student strategy in this way provided an opportunity for each teacher to consider what details were important to attend to that might indicate the student's potential understandings.

Patterns of interaction: Independent construction. While I identified interactions that encouraged teachers to construct children's strategies together, most episodes contained interactions that I considered as low or no contributions, or interactions that facilitated one teacher constructing the student's strategy. I characterized these instances as interactions where the partner teachers indicated they were following, and possibly understanding, what the speaking teacher noticed, but did not contribute a new idea to what was noticed. Interacting in this way generally sounded like the partner teacher (a) agreed with the descriptions or claims the speaking teacher shared, (b) finished a sentence or repeated a phrase, or (c) asked a clarifying question. When teachers engaged with one another in this way, the speaking teacher had an opportunity to notice the student's mathematical thinking while the partner teacher demonstrated confirmations, but did not contribute to the idea. Therefore, I considered interactions like these as opportunities for each teacher to independently construct the individual child's thinking rather than joint construction.

Opportunities for the speaking teacher to reflect and refine. Within the episodes there was a subset of interactions that gave the speaking teacher the opportunity to pause and reflect on or refine their ideas. These moves often functioned as a partner teacher clarifying a detail or claim by asking a question. For example, referring back to Damien's strategy (Figure 15), Molly asked Janice "Did [Damien] have that before you, or you had to help him because he had no way to start the story?" When Molly asked this question, Janice had an opportunity to reflect on what she knew about Damien's understanding of a half; however, this type of clarifying question did not demonstrate Molly was noticing in this turn.

Instances like these are important to note because they indicate potential opportunities for the teachers to make sense of student thinking together. In order for the partner teacher to engage in conversational moves that helped the speaking teacher to reflect or refine his or her noticing, the partner teacher may have also noticed something about the student's mathematical thinking that she or he wanted to ask about. However, due to the nature of these interactions, the partner teacher did not contribute a new idea to what was noticed about the student's mathematical thinking, but rather positioned the speaking teacher to further elaborate on their own noticing.

Opportunities for the speaking teacher to make their own noticing visible. I identified slightly less than one-third of the episodes as containing no exchanges related to noticing children's mathematical thinking. These episodes provided an opportunity for one teacher to make what he or she noticed children's mathematical thinking visible, but not an opportunity for both teachers to make sense of the child's mathematical thinking together. Within these episodes, partner teachers were rarely, if ever, silent during the interaction. Partner teachers usually indicated they were following, and possibly understanding, what the speaking teacher noticed. However, the partner teacher often responded with sounds such as “uh-huh” or “hmm,” or small phrases such as “interesting,” or “right,” making the interactions largely one-sided and difficult to interpret what the listener took away from the interaction.

Opportunities for partner teacher to make noticing visible. In addition, there were a small number of episodes in this category that contained instances where the partner teacher attempted to either introduce or respond to an idea; however, the speaking teacher did not relinquish their turn. Therefore, a noticing exchange was not identified. I noted these interactions, but did not analyze these interactions further, because it is difficult to interpret how these instances of listener contributions could have contributed to the teachers’ working together to collectively notice children’s mathematical thinking, as the speaking teacher did not take up or respond to this interaction.

Accounts. While I reviewed the episodes, I also noted a particular type of interaction where teachers retold accounts of a conversation they had with the student. Often accounts were one long turn providing elaborate descriptions. These types of accounts should be expected because when teachers downloaded the problem from the collaborative inquiry tool, there was a prompt that suggested they ask their students questions during the problem-solving task. However, because I was interested in how teachers worked together to collectively notice children’s mathematical thinking, I highlighted these types of episodes in particular to better understand in what ways teachers had an opportunity to collectively notice children’s mathematical thinking. I identified 33 episodes of teacher accounts, or descriptions of the shared interactions one teacher had with a student. Within these episodes, I identified nine episodes with no contributions and the remaining episodes contained at least one interaction related to noticing.

Consider an episode between Jill and Melissa as they discussed James' strategy (Figure 21) for a multiple groups measurement division problem, the number of 1/2 cup servings in three cups. James used a direct modeling strategy representing the cups of frog food as rectangles and partitioning each rectangle by the serving measure and counting up the number of servings. There are additional equations and inequalities on the page that Jill posed to James while she talked with him about his strategy and elicited his understanding.

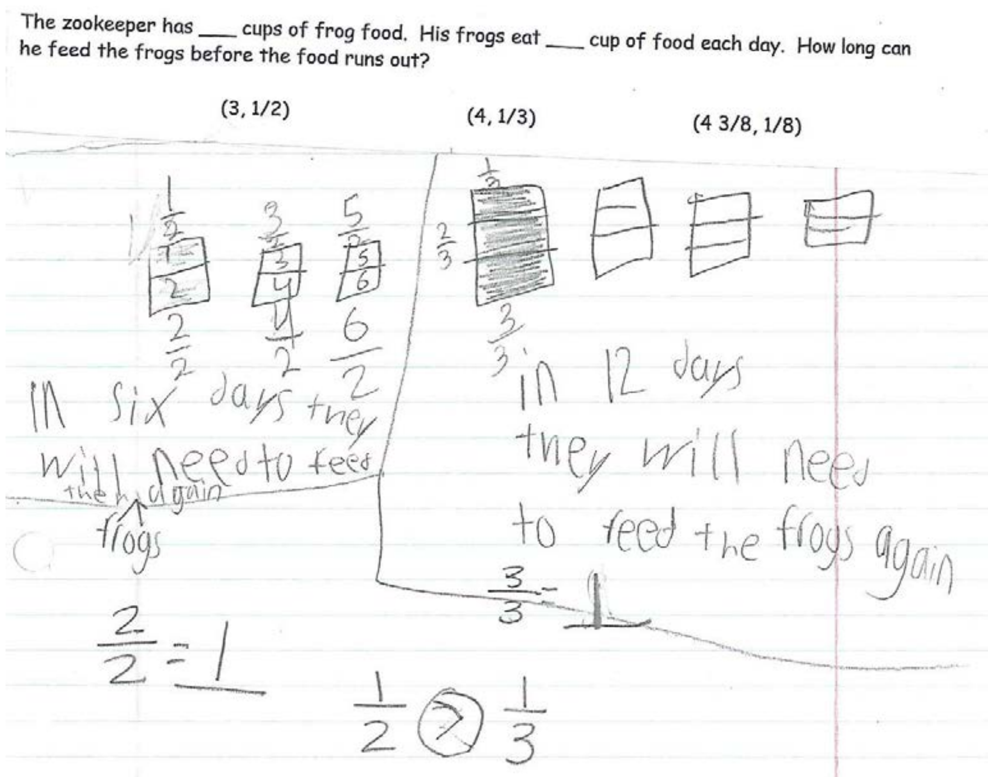

Figure 21. Image of James' Strategy

Transcript 10: James' Strategy

1. JILL: So, um. Okay, so what James did is he drew his cups. Three cups of food, frog food. And he was able to divide his cups into halves and he numbered them one two three four five six to get his answer and he had six days. That's all he had. And then when I came around to him, um I was like, well, can you show me what is this one? Is that one cup? I kind of made it like I didn't know what the one…. What is that one? I don't-, I'm not un-, I'm not sure what that one is. And so he had to explain that that was the amount that they ate in one day. And so I said how much was that amount? And he said half and I said, well can you write that there? And I said, he was able to write that there. And so I said, so that's-, so how much is that then? This whole thing? And he said, well, that

would be two halves. And then we went over here-

2. MELISSA: And he labeled it correctly, I mean

3. JILL: He did

4. MELISSA: Two halves [2/2]

5. JILL: He did, which I was surprised. I was not expecting him to label it as three-halves, four-halves, five-halves, six-halves. So I just started going over here and asking him about, um, two-halves. And I said, I did this little extension and I said, well, what is two-halves, what could two-halves also equal? And he, he said one whole. I think I might have asked him if that was true or not. And, um, he said that was true. Later on, we went into, when we did this one, same thing. He started labeling this one. He did the same thing, he just drew it, counted up. And he was done, 12. And so, you can see, I went back and I was like okay so what is this part? And so he did the same thing, one-half, two-thirds, oh, one-third, two-thirds, three-thirds. So I questioned him again about that and then once we were, you know, we were talking. I asked him, well which one do you think is more? I just wanted to see if he could figure it out. And he knew exactly, he said well, one-half is gonna be more. First he said one-third, and I said really? And he said, no no no no. It's because one-half you have less, less numbers, less parts, so you're gonna have more. So one-half is gonna be more. And he came to that all on his own.

6. MELISSA: Well good.

7. JILL: Yeah. So, I was real pleased with that.

8. MELISSA: Good.

9. JILL: Definitely using this, you know, I'm able to kind of walk around and get kids thinking more.

10. MELISSA: Extending, yeah.

11. JILL: Yeah. And extending it more.

Within this episode, Jill shared what she noticed about James' mathematical thinking (Transcript 10); however, Melissa took a few turns that showed some engagement with noticing. For example, after Jill finished describing James' first strategy, as she began to shift to his second strategy, Melissa interrupted at Turns 6 and 8 to state James used symbolic notation, labeling his halves. Jill responded to this idea in Turn 9 by elaborating on this detail, Jill counted each half and described how she posed an extension equation during the problem-solving task, asking James what number is equivalent to 2/2. While the noticing exchange between the two teachers is considered to be a form of high contributions, no additional contributions were identified in this episode.

I characterized most episodes that contained an account as no contributions, potentially constraining opportunities for the teachers to work together to collectively notice children's mathematical thinking. The three episodes that I characterized as an account with at least two instances of high contributions came from three unique groups, and included the interaction between Debra and Claudia in Transcript 7 about Dillon's strategy, suggesting there could be opportunities for teachers to share accounts from their classroom and work together to collectively notice children's mathematical thinking with their partner teacher, opening up opportunities for collective noticing.

Summary of the patterns of interactions. There were characteristics across the episodes that both opened and constrained opportunities to collectively notice children's mathematical thinking. While most episodes contained opportunities for one teacher to make his or her noticing of children's mathematical thinking visible, teachers did create opportunities to jointly construct children's mathematical thinking in collective noticing by interacting in ways that allowed both teachers to contribute their own noticing. This data suggests teachers may have some implicit norms for negotiating turn-taking that allow the partner teacher to verbalize his or her own noticing of the presented student's mathematical thinking.

In Figure 22, each of the 29 sessions that I analyzed is represented along the horizontal axis. Within each session, the number of episodes that were coded as demonstrating episodes with high, low, and no contributions are shown with blue, green, or yellow bars, respectively.

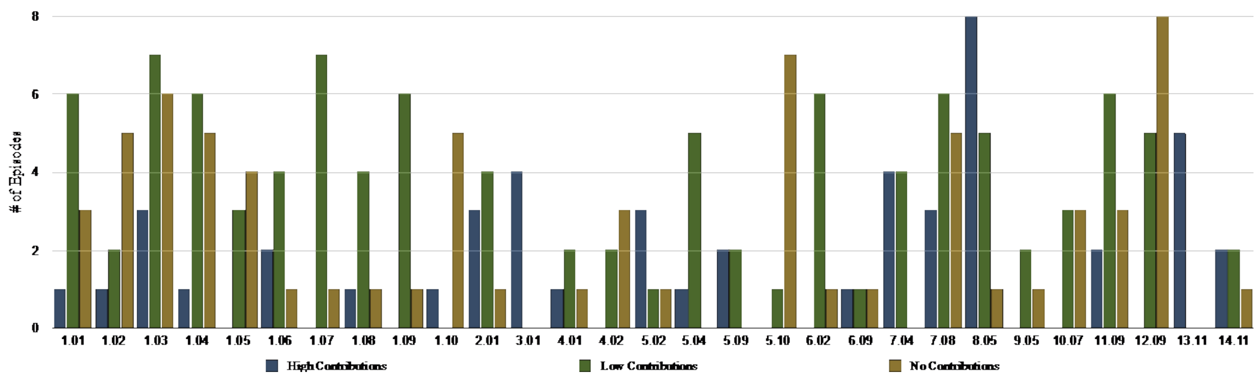

Figure 22. Patterns of Interaction by Session

All 29 sessions included at least one episode with a conversational exchange grounded in the details of the student's strategies and two sessions included only contributions that were characterized as high. This finding suggests that partner teachers engaged in collective noticing may naturally find ways, without a facilitator, to contribute to an idea related to noticing, although the level of the contribution may vary. In addition, the two sessions that only included episodes in which the teachers jointly constructed students' strategies. Identifying groups that mostly engage in joint constructions of students' mathematical thinking could serve as focal case studies for future studies to investigate characteristics of groups that might contribute to these patterns.

One-fifth of the sessions included more episodes with no contributions than episodes with at least one contribution related to noticing, or the partner teacher either did not take a conversational turn, or the group did not take up their contribution. This finding suggests teacher groups have the capacity to engage with one another in ways that could potentially facilitate the joint construction of collective noticing of children's mathematical thinking, but there could be constraints within the group or the module that limit this engagement. In the next section I will present episodes showing the possible associations between the teachers' quality of noticing and their conversational interactions.

The collective noticing of children's mathematical thinking

Returning to my research question, I wondered how teachers collectively engaged in the practice of noticing children's mathematical thinking. Here I combine the findings from the analyses of the quality of collective noticing with findings from patterns of teachers' interactions to consider how the discussion was productive for teachers, or how it opened up opportunities for collective noticing.

Looking across the episodes (see Table 5), both the quality of collective noticing and patterns of interaction, most episodes demonstrated teachers independently constructing the students' strategies, or roughly half of the episodes. However, when teachers jointly constructed students' strategies, there were fewer instances of the teachers demonstrating a lack of evidence of noticing children's mathematical thinking than when they independently constructed student strategies. This suggests that teachers meeting together to collectively notice children's mathematical thinking is a productive activity for teachers as their discussion around student work are mostly grounded in the details of student strategies. Furthermore, when teachers' discussions demonstrated instances of higher contributions, in which both teachers contributed to the noticing, they potentially created an opportunity to remain grounded in the details of the student strategy, as the teachers requested details through their elaborations, claims, and counterclaims.

Table 5

Quality of and Patterns of Interaction in Collective Noticing

	Joint Construction	**Independent Construction**	
	High contributions	Low contributions	No contributions
Robust evidence of collective noticing of children's mathematical thinking	29	29	11
Limited evidence of collective noticing of children's mathematical thinking	18	48	37
Lack of evidence of collective noticing of children's mathematical thinking	2	25	21

Debra and Claudia jointly construct student strategies in collective noticing. Teachers engaging in a joint construction of collective noticing could potentially facilitate discussions that remain grounded in the details of student strategies, it is important to look closely at episodes where this pattern emerges over many episodes within a session. Interactions that demonstrated joint construction of collectively noticing children's mathematical thinking can be exemplified through a session with Debra and Claudia. At the time of the data collection, Debra and Claudia were 4th and 5th grade teachers from the same school. The teachers were in their second year of the Extending Children's Mathematics professional development and were completing their last collaborative inquiry session for the year. In this particular session, the problem the teachers were discussing was an equal sharing problem, a familiar problem type; however, the module encouraged teachers to include a mixed number as the number of items, requiring students to share a fractional amount, or an item that was not whole. In the session, Debra and Claudia discussed 14 pieces of student work for about 30 minutes. Episode lengths within this session varied from 32 seconds to 5 minutes and 20 seconds.

Within the session (see Figure 16, Session 8.05), three episodes were characterized as providing robust evidence of noticing children's mathematical thinking, 11 episodes were characterized as providing limited evidence of noticing children's mathematical thinking, and no episodes were characterized as providing a lack of evidence of noticing children's mathematical thinking.

Reviewing the teachers' patterns of interactions, Debra and Claudia engaged in some form of contribution for all but one episode and appeared to work together to make sense of student thinking in more than half of the episodes. While most of the episodes demonstrated limited evidence of noticing children's mathematical thinking, the teachers engaged in both high and low contributions for all but one episode. The teachers' conversations were grounded in the details of the students' strategies, and there were several interactions that encouraged the teachers to sustain in the details. Recall the three episodes previously mentioned: Avery (Figure 17), Dillon (Figure 18), and Conner (Figure 20). In these episodes the teachers elaborated on one another's details and offered alternative claims and perspectives, opening up opportunities for the teachers to consider a new way to interpret the student's strategy or to determine what details are important for understanding the student's strategy.

There were also a few episodes where the student work the teachers discussed was unclear, but through collective noticing, the teachers persisted discussing the student's written work. For example, consider Claudia and Debra's discussion about Jeremy's strategy for the problem 2 share 6 1/2 (Figure 23, Transcript 11). Jeremy drew two kids and seven rectangles. Most likely, the six large rectangles represent the six whole sandwiches, and the small rectangle represents the half sandwich, but it is not clear how Jeremy partitioned the rectangles or how he may have distributed the sandwich partitions. When Claudia claims Jeremy cut the sandwiches into sixths (Turn 1), Debra looked carefully at the rectangles Jeremy had drawn and asked if the student created sixths or ninths (Turn 2). Jeremy appeared to draw two vertical lines that could represent a partition into thirds. He also drew a few darkened horizontal lines in the middle of the rectangle, which could represent either partitions or the peanut butter filling from the problem context. However, Claudia suggested the student partitioned into sixths and that the student did not understand the problem (Turn 3), most likely interpreting the dark horizontal line as an initial partition into half, and then two vertical lines as a further partition into sixths. Thinking the student misunderstood the problem, Claudia then discussed how she might respond to the student (Turn 5), but Debra called attention to the drawing again as a way to suggest Jeremy understood the problem (Turn 4). As Claudia began to describe again how she might respond to Jeremy (Turn 5), Debra called attention to Jeremy's answer, wondering if he meant another number when he wrote "2 quarters." Claudia considered another way to interpret the student's answer (Turn 9), but decided the answer would still not be valid.

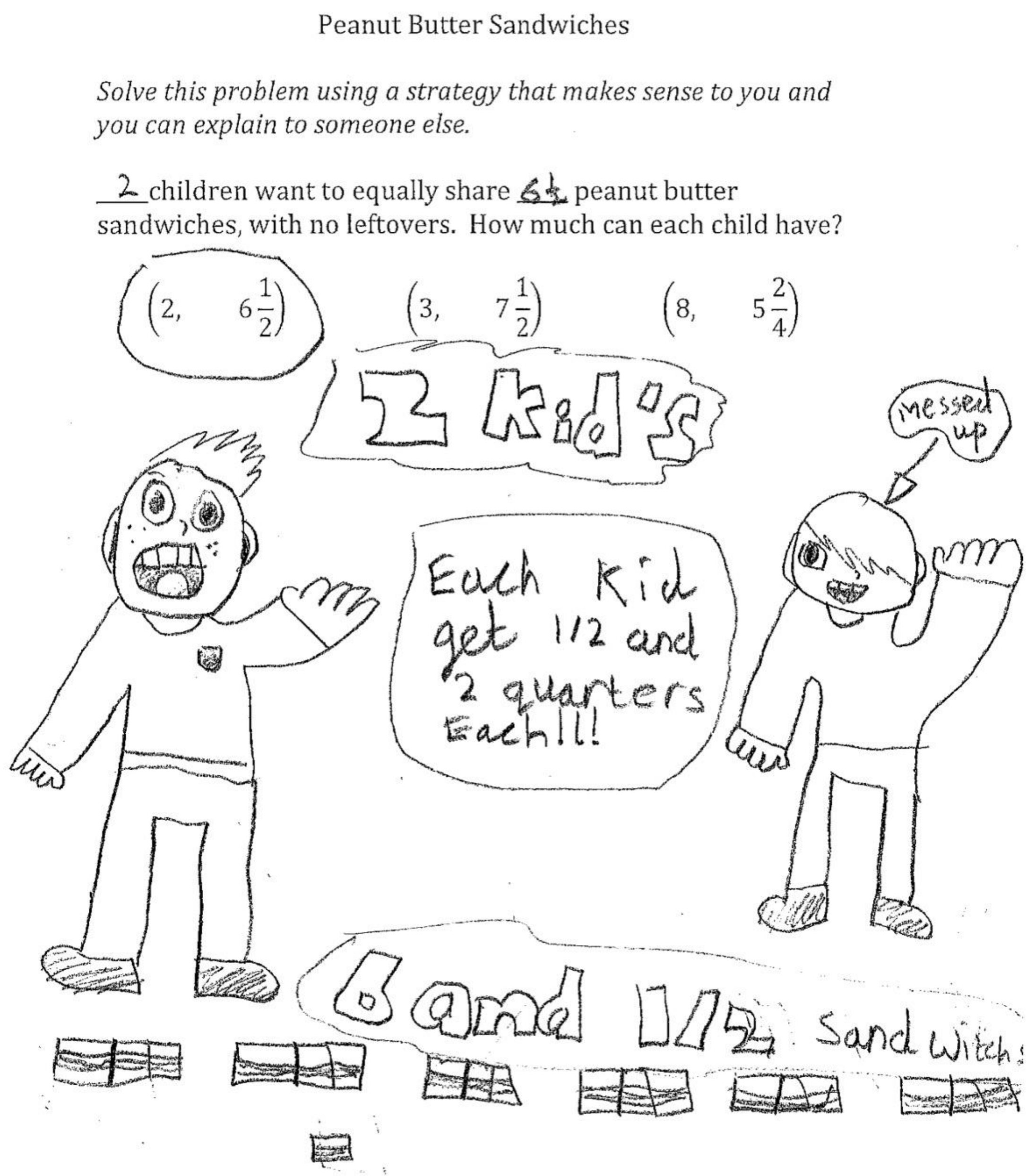

Figure 23: Image of Jeremy's Strategy

Transcript 11: Jeremy's Strategy

1. CLAUDIA: The problem was, two children wanna share six-and-a-half peanut butter sandwiches. He's got two children that he drew. His answer is each kid gets

one-half and two-quarters each. When we look at that, though, it’s not represented. He’s cutting them into sixths.

2. DEBRA: He has the six—he has six sandwiches, and he cut them into six pieces each. What is that? Oh, are those his lines? Are you sure he cut them into sixths, or ninths? I don’t know.

3. CLAUDIA: I think it’s sixths. He cut it into sixths. Again, I don’t know if he misunderstood this problem. With him, I would go back and talk with Jeremy about, if we’re sharing it with two people, do we need to cut it into six pieces?

4. DEBRA: Cuz he clearly has two kids, and he clearly has six-and-a-half sandwiches, so he does know the two kids and the six-and-a-half sandwiches. Okay, go ahead.

5. CLAUDIA: I think that he cut them in half, but I don’t think that he was understanding—I don’t think he was understanding how they were put together. I think he was confusing the wording in the problem, for some reason, and I’m not sure why. I need to back up with him and say, “Okay, if each kid gets one-half and two-quarters, can you show me what that looks like,” so that when he sees that would make up one sandwich. Then I would go back to, well how many sandwiches do we have? Six-and-a-half. Have we shared them all? No. I’m hoping that guiding him in that way, he would see that.

6. DEBRA: Yeah, it makes me wonder, too, if he just messed—if he just messed up and didn't mean quarters, but meant sandwiches.

7. CLAUDIA: Maybe.

8. DEBRA: I don't know.

9. CLAUDIA: Because he is from [another English-speaking country], too, and so, sometimes there is a language barrier there with certain things that he says. Or, he may just say, "Oh, I meant to write." Even so, even if he did that, that still wouldn't be enough. Even if he meant two sandwiches, that would mean four sandwiches, plus the half would be five sandwiches.

10. DEBRA: Oh, yeah, that's right.

11. CLAUDIA: We'd still be missing one-and-a-half sandwiches.

12. DEBRA: Yeah, true.

13. CLAUDIA: I think, with him, I need to go back and talk with him about, "Okay, can you explain this to me first?" See what he's thinking, and then walk him through, like I said, why did we cut this up into sixths? Do we need to—do two kids need to cut into six pieces? Things like that. See if maybe he can get to the half, and then have him see if that answer, does it make sense. That's a big thing I'm sure you work on.

14. DEBRA: Recently.

15. CLAUDIA: Yes, does it make sense?

While this episode may not have been productive for the teachers to fully understand how the student solved the problem—as the details of the student's strategy were unclear—by engaging in collective noticing, the teachers created an opportunity to reconsider what the details represented and how the child solved the problem. In addition, through the renegotiation of the details, Claudia also took an opportunity to modify how she might respond to Jeremy. In Turns 3 and 5, Claudia shared some directive actions that could take over the child's thinking, but in Turn 13, Claudia suggested she could begin her conversation with Jeremy by first asking him to explain his strategy, providing an opportunity for Jeremy to share how he thought about solving the problem, before moving to more corrective directives. Therefore, when teachers jointly construct student strategies, they engage in productive moves that remain grounded in the details of student strategies, contributing to their collective noticing.

Summary

In this chapter, I presented examples to demonstrate the quality of and interactional patterns for collective noticing in self-facilitated sessions. The sessions were designed to provide an opportunity for all teachers to engage in collective noticing; however, this opportunity was not taken up in the same way across or within groups. Findings suggested that teachers were able to discuss details of a child's problem-solving strategy, remaining grounded in the details of students' strategies. However, the types of contributions provided teachers with the opportunity to make sense of the strategy either together or independently. In the last chapter, I will further discuss the findings, implications, and limitations of this study, as well as possible future directions.

Chapter 6 : Discussion

This dissertation explored the opportunities 3rd-5th grade teachers created to collectively notice children's mathematical thinking as they interacted in self-facilitated collaborative inquiry groups. I analyzed audio recordings of teachers' discussing written student work from their classrooms to determine how teachers' discussions were grounded in the details of their students' strategies and their patterns of interaction in collective noticing.

The central question for my study was *how do teachers collectively notice children's mathematical thinking when participating in self-facilitated collaborative inquiry?* I found that when teachers discussed written student work with a colleague, they mostly remained grounded in the details of the students' strategies and occasionally interacted in ways that contributed to a joint construction of the student strategy.

Considering the quality of the collective noticing, most discussions remained grounded in the details of student strategies without the presence of an additional facilitator. Many researchers have documented the importance of facilitators in encouraging teachers to remain focused on the details of a student strategy and pressing teachers to elaborate on the details (Amador & Carter, 2018; Andrews-Larson, Wilson, & Larbi-Cherif, 2017; Little, 2003). However, my findings suggest that teachers participating in self-facilitated collective-inquiry not only have the potential to take on the facilitation role, but can also take an opportunity to jointly construct a student strategy. This joint construction of student strategies could potentially facilitate more complete descriptions of student thinking. While there were instances in which a student's teacher began to use prior knowledge about their students to examine student work

(Goldsmith & Seago, 2011), discussions of what the child could not or did not do (Horn, 2005; Louie, 2015), and the correctness of the student's answer (Krebs, 2005), describing this work as a collective created opportunity for claims to be supported by the details of the strategy.

Considering the patterns of interactions, my findings suggested the persistence of individual contributions within the collective noticing; however there were teacher groups that interacted in ways that allowed for teachers to jointly construct student strategies, engaging in noticing. While the web-based protocol did not include prompts that might encourage partner teachers to ask probing questions or contribute their own noticing, teachers may have invited the partner teacher to provide his or her own perspective. In addition, while the data did suggest instances where partner teachers provided counterclaims, they made up a small number of the contributions, a finding also confirmed by Chamberlin (2005). This indicates the complexity of these types of interactions, which may both require and lead to a deeper understanding of the student's strategy. Therefore, although not confirmed through the data analysis, as teacher noticing of children's mathematical thinking improves, and in particular considering alternative perspectives and possibilities, opportunities to jointly construct children's mathematical thinking may be created, leading to more robust descriptions of children's thinking as revealed by mathematical strategies.

Horn and Kane (2015) asked if self-facilitated teacher collaborations are productive for teachers who are in the process of developing a practice. Teaching experience alone does not promote expertise in noticing (Dreher & Kuntze, 2015; Jacobs et at., 2010); but rather teachers must engage with research-based frameworks on children's mathematical thinking as they elicit the mathematical thinking of their students. When teachers participate in discussions that are grounded in the details of student strategies, they have an opportunity to continue to develop expertise in their noticing of children's mathematical thinking through the articulation and reflection of children's mathematical thinking (vanEs & Sherin, 2008).

My findings suggest that teachers participating in a professional development centered on children's thinking were able to self-facilitate discussions that are grounded in the details of student strategies, even as they participated in this type of professional development for the first time. The teachers were encouraged to meet for the single purpose of collectively noticing children's mathematical thinking and used a protocol which encouraged teachers to select student work that demonstrated a range of understanding and discuss each strategy separately.

Little and Curry (2009) argue that protocols are a limited resource for structuring conversations that promote discussions of teaching and learning. An additional support that may have contributed to teacher discussions of student work was their participation in a minimum of 8.5 days of professional development focused on introducing research-based frameworks of children's mathematical thinking. During the professional development, teachers had many opportunities to review and discuss videos of children solving problem and representations of student thinking with participant teachers and a professional development facilitator. These discussions of children's thinking began with describing the details of the strategy and interpreting potential understandings as revealed by the mathematical strategy. Participating in these discussions over many weeks could have established tacit norms that structured teachers' conversations around the mathematical details of their students' work.

Limitations

Not all collaborative inquiry groups created as part of the Extending Children's Mathematics professional development were included in the final analysis, and one group made up roughly a third of the collected sessions. Moreover, there is some selection bias as teachers opted in to the study by submitting audio recordings. For this reason, while I reported findings, they should not be read as either typical or representative of all collaborative inquiry groups.

Future Directions

Looking forward, there are a few additional features to consider when exploring how the teachers collectively noticed the mathematical thinking of their students.

The analysis in this dissertation study considered the episodes of discussing children's written work as isolated instances of discussion. However, these episodes did not exist in isolation, but rather were embedded in a session. Therefore, as a next step, it would be important to reconsider the episodes within a session as a sequence of episodes. In re-approaching the data in this way, it may demonstrate how teachers may make connections across pieces of student work and may serve as a better estimate for how the teachers discussed the mathematical details of their students not originally considered. For example, many student strategies were characterized as direct modeling strategies. Perhaps, as teachers moved through the conversation, some of the details were no longer made explicit because the details were similar enough to other pieces of student work and the teachers did not feel the need to repeat that piece of the strategy. Therefore, in instances like this, discussions of the details may have been more robust than identified in this study.

Another feature to consider are the characteristics of student strategies that may facilitate more productive conversations. Students can produce strategies that clearly indicate their problem-solving process and teachers may be able to accurately describe the details of the strategy, but the work students turn in is not always clear. For example, students could use atypical notation or strategies, their markings may not follow conventional writing structures (starting at the top-left of the page), or students may use a mental strategy. Therefore, it is important to consider how teachers navigate these features and in what ways they can either open or constrain opportunities for teachers to collectively engage in noticing children's mathematical thinking

Lastly, an important feature to consider further are the patterns of interactions teachers seemed to engage in as they discussed the written work of their students. I made claims that the joint-construction of noticing children's mathematical thinking could perhaps lead to more robust discussions; however most of the interactions seemed to demonstrate a more one-sided conversation, with one teacher describing the details of the student strategy. For example, I considered interactions where partner teachers agreed with or repeated the noticing of the presenting teacher to be low-contribution and supportive; however Crespo (2006) discussed how repetitions demonstrated intellectual engagement among the group. In addition, I considered conversational moves in which the partner teacher asked clarifying questions as a form of low-contribution because the description of the details continued to be refined by the initial teacher. Therefore, further investigation should continue to consider the role of these interactional patterns as teachers engage in collective noticing, in particular in instances with no or low contribution.

Furthermore, it would also be important to investigate how forms of contributions are related to characteristics of the group and their capacity to collectively notice. As my study seemed to suggest an association between high contribution patterns and teachers presenting more robust evidence of collectively noticing children's mathematical thinking, it is important to consider under what conditions higher contributions occur and how might these interactions be fostered among different groups to facilitate the joint-construction of collective noticing.

Conclusions

My study asked how teachers collectively notice children's mathematical thinking in self-facilitated discussions. I responded to this question by analyzing teacher discussions for the quality of their collective noticing of children's mathematical thinking and the patterns of interaction that perhaps facilitated their collective noticing. I conclude that collaborative inquiry groups for the purpose of collective noticing may help teachers self-facilitate conversations that are grounded in the details of students' strategies and potentially encourage teachers to work together as their conversation is anchored to a student's strategy that both teachers can see. Furthermore, collective noticing is potentially enhanced when teachers jointly construct, or both make contributions to the description and interpretation of, student's mathematical thinking because both teachers make their noticing visible to both themselves and to one another.

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