TEACHERS' KNOWLEDGE OF CHILDREN'S STRATEGIES FOR EQUAL SHARING FRACTION PROBLEMS

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In this exploratory study, we documented teachers' knowledge of children's mathematical thinking as they engaged in the task of anticipating children's strategies for an equal sharing fraction problem. To elicit an array of knowledge, 18 teachers were deliberately selected with a variety of numbers of years participating in professional development focused on children's mathematical thinking. We characterized the flexibility of teachers' knowledge by the number of valid and distinct strategies teachers anticipated and the degree to which these strategies reflected strategies research has shown children typically use. We argue that this flexibility is necessary for teachers to engage in instructional practices that are responsive to children's mathematical thinking.

Keywords: Teacher Knowledge, Mathematical Knowledge for Teaching, Rational Numbers, Elementary School Education

Introduction

In this study we explored the knowledge that 18 upper elementary teachers displayed as they engaged in the task of anticipating children's strategies for the following equal sharing fraction problem: *A teacher has 4 pancakes to share equally among 6 children. How much pancake does each child get?* To elicit an array of knowledge, we selected for our analysis teachers with a range of numbers of years of participation in professional development focused on children's mathematical thinking. Our goal was to characterize teachers' knowledge of children's mathematical thinking in terms of its flexibility.

We view *knowledge of children's mathematical thinking* as a distinctive type of knowledge that is essential for teaching in ways that are responsive to children's mathematical thinking (Jacobs & Empson, 2016). We take the perspective that teachers' knowledge is situated (Putnam & Borko, 2000) and thus are interested in the knowledge teachers use as they engage in instructional practices that are part of responsive teaching. We focused specifically on the instructional practice of anticipating strategies, adopting what Stein, Engle, Smith, and Hughes (2008) defined as teachers' consideration of the array of strategies students would be likely to use for a given problem and the potential mathematics that could be learned with those strategies. In this study, we focus on teachers' anticipation of the array of strategies students would be likely to use.

Our analysis of teachers' knowledge was informed by the idea of flexibility. We considered the flexibility of teachers' knowledge in two ways. First, we used Whitacre's (2015) focus on the number of distinct and valid strategies that a teacher anticipated. Second, we considered to what degree did the teacher's anticipated strategies provide evidence of knowledge that was organized primarily around children's ways of making sense of the mathematics or around the teacher's own ways? In other words, to what extent was a teacher able to shift perspective and "think like a child"?

This study represents an exploratory effort to document teachers' knowledge of children's mathematical thinking in the domain of fractions as it was used to anticipate children's strategies for an equal sharing fraction problem. Teaching is inherently knowledge intensive. Documenting the

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knowledge teachers use in instructional practices such as anticipating strategies is an essential step in promoting the development of expertise in teaching responsively on the basis of children's mathematical thinking.

Conceptual Framework

Teachers' knowledge of students and content has been conceptualized as a type of pedagogical content knowledge involving knowledge of how students learn a specific topic (Hill, Ball, & Schilling, 2008). It includes knowledge of students' strategies, common misconceptions, and typical developmental paths. Although its importance is widely acknowledged, teachers' knowledge of students and content has been the explicit focus of only a handful of studies (e.g., Bell, Wilson, Higgins, and McCoach, 2010).

In the current study, we focused on teachers' knowledge of students' strategies. We asked teachers to anticipate the range of strategies children might use to solve a story problem involving fractions and then to identify which strategies showed the most basic and advanced understanding. We then analyzed these strategies to explore the flexibility of teachers' knowledge. We were interested in two types of flexibility. The first type of flexibility involved the number of valid and distinct strategies anticipated by a teacher (Whitacre, 2015). We defined strategies as *valid* if a teacher's written response provided evidence about a process a child could use to solve the problem and reach a correct answer. Evidence of a process (vs. only an answer) was important because we wanted to elicit teachers' knowledge of children's thinking and not simply whether or not a teacher was able to solve the problem; if a teacher provided only an answer (e.g., "4/6 = 2/3"), the response was not counted as a valid strategy because we could not tell what reasoning the teacher thought the child might use to arrive at that answer. Strategies were further defined as *distinct* if they involved a qualitative difference in the process a child might use (e.g., partition pancakes into sixths vs. partition pancakes into thirds). Strategies were not considered distinct if they differed only in terms of a superficial feature that did not reflect a different process (e.g., partition into sixths and distribute pieces by numbering the pieces vs. partition into sixths and distribute pieces by drawing lines to connect to sharers).

The second type of flexibility involved the extent to which a teacher's anticipated strategies were organized around categories of strategies that children typically use, as documented in past research (Empson & Levi, 2011). The more flexible a teacher's knowledge is in this sense, the more the teacher is able to "think like a child." The less flexible in this sense, the less a teacher is able to see the problem the way a child might. Instead, teachers with less of this second type of flexibility may tend to anticipate strategies that are organized around instructed procedures and conventions—generalized, all-purpose methods that are introduced in school mathematics. Although the use of instructed procedures and conventions is a valuable goal of school mathematics it does not necessarily signify understanding on the part of the child or link to the child's informal strategies. In particular, when procedures and conventions are explicitly taught before children have had a chance to advance their understanding of fractional quantities, children's use of instructed strategies tends to be error-prone and procedurally-driven. Teachers who anticipate strategies mainly in terms of instructed procedures and conventions may be unaware of the role of children's informal strategies in the advancement of children's understanding of fractions.

These two types of flexibility provide a basis for describing the knowledge that enables teachers to be responsive to children's mathematical thinking: the greater the flexibility of teachers' knowledge, the greater their ability to make sense of and respond to children's ever evolving thinking. We therefore identified in teachers' anticipated strategies the degree to which there was a focus on children's typical strategies and noted how instructed procedures and conventions were used.

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Focus on children's typical strategies. Children's strategies for equal sharing have been well documented (Empson & Levi, 2011). When invited to solve problems on the basis of what makes sense to them, children use a variety of informal strategies that are driven by their understanding of partitioning and equal distribution to create fractional quantities. There are three main categories of strategies typically used by children to share 4 pancakes among 6 children, distinguished by the progressive abstraction of fractional quantities, which reflect advances in understanding. In the most basic category, children draw models of some sort – often circles or rectangles – and partition them using familiar or easy-to-make partitions such as halves and fourths without consideration for the number of sharers. Pieces are distributed one by one to sharers. This category is called *non-anticipatory direct modeling*. As children's understanding of fractions develops, they continue to draw models and distribute pieces one-by-one, but they partition the models in a manner linked to the number of sharers. If there are six sharers, a child might begin by partitioning one pancake into sixths or two pancakes each into thirds. In both cases, six pieces are created, one for each sharer. This category is called *emergent anticipatory direct modeling*. In the most advanced category of strategies, children operate on mental models rather than drawn models; in particular they understand and are able to use the relationship that one pancake divided among six children is equivalent to one-sixth pancake per child in a variety of ways to figure how much pancake each child gets. They may use notation to support their thinking and express the strategy as $1/6 + 1/6 + 1/6 + 1/6 = 4/6$, $4 \times 1/6 = 4/6$, or $4 \div 6 = 4/6$. This category is referred to as *anticipatory*. Children may also use various combinations and transitional versions of these three main categories as their understanding develops.

Focus on instructed procedures and conventions. Children may also use instructed procedures and conventions while solving an equal sharing fraction problem. For example, they may "invert and multiply" ($4 \div 6 = 4/1 \times 1/6 = 4/6$), simplify fractions to lowest terms ($4/6 = 2/3$), or use long division. Children may use such procedures and conventions because their understanding of fractions has reached a level of fluency in which operations are routine and do not need to be directly modeled or decomposed into simpler computations, as in the informal strategies above. However, children also use such procedures and conventions because they think it is expected rather than because they understand fraction operations well. A procedure such as long division with a repeating decimal answer, for example, is an inefficient choice of strategy for this problem because of the number of steps involved and the difficulty of relating the answer to a fractional size of cake.

Methods

Data and Participants

The data analyzed for this study were taken from a sample of 71 teacher written responses collected to pilot an assessment of elementary teachers' knowledge of children's mathematical thinking about fractions. The teachers were all teaching grade 3, 4, or 5 at the time and reported a number of years of participation in professional development focused on children's mathematical thinking ranging from no participation to three or more years. In selecting our sample our goal was to maximize to the extent possible the variation in teachers' responses with respect to knowledge of children's mathematical thinking. We selected all teachers who reported 3 or more years of professional development focused on children's mathematical thinking and all teachers who reported no years of such PD. This selection resulted in a sample of 18 teachers (9 from each end of the range).

Task

A teacher gave this problem to the class: A teacher has 4 pancakes to share equally among 6 children. How much pancake does each child get?

a. *Provide 5 different valid strategies that represent the range of strategies that elementary students might use to solve this problem. Write out or draw each strategy the way a student might.*
b. *Mark the strategy that indicates the strongest understanding of fractions.*
c. *Mark the strategy that indicates the most basic understanding of fractions.*

Figure 1. The task posed to teachers in the study.

Teachers were given the task in Figure 1 to elicit their ability to anticipate a range of strategies and differentiate the understandings reflected in those strategies. We recognize that the written nature of the data limited the extent to which we could explore teachers' knowledge of children's mathematical thinking captured by this task. In particular, because we could not ask follow-up questions, we did not have in-depth information about the reasoning behind their choices.

Analysis

We began by coding each teacher's set of anticipated strategies for our two types of flexibility: (a) counting the number of distinct and valid strategies and (b) determining the degree to which the teacher's focus was on children's typical strategies. We also identified the specific strategies anticipated by each teacher to get a sense of his or her range focused on children's typical strategies and examined the details of these strategies for consistency with children's reasoning. We also coded each teacher's ability to differentiate the most and least understanding in the strategies. Finally, we synthesized all of this information to develop profiles that reflected teachers' knowledge and its flexibility. Three of the authors double-coded the data, with discussion and resolution of discrepancies.

Findings and Discussion

Teachers' knowledge of children's mathematical thinking as reflected in their anticipated strategies for equal sharing fell into three groups. We discovered that the number of distinct and valid strategies did not differentiate meaningfully between groups of teachers, and so in our profiles we focused on the degree to which teachers' anticipated strategies were consistent with children's typical strategies, both in terms of which strategies were used and the details within those strategies.

Profile 1: Robust evidence of knowledge of children's mathematical thinking

Each teacher in this group anticipated a set of three or more strategies in which the strategies and the details of the strategies were consistent with children's typical strategies. Teachers were also consistent in their assignment of least and most advanced understanding. Overall, these responses reflected teachers' ability to "think like a child." Figure 2 shows a set of these strategies from one teacher in which all of the strategies are consistent with the kinds of strategies children might use, as documented in research. The first strategy, non-anticipatory direct modeling, involved the use of repeated halving and did not exhaust the pancakes. The next strategy, emergent anticipatory direct modeling, involved the use of drawn models to partition the pancakes and then distribute the pieces one by one (by numbering them). The pancakes were partitioned into sixths — a common partition that children use for six sharers — and the final answer in the strategy was expressed in a manner generally consistent with how children might express it. The third strategy, another example of emergent anticipatory direct modeling, involved partitioning all four drawn pancakes into thirds, also a common partition that children use. The fourth, more advanced transitional anticipatory strategy used a drawn model to partition two pancakes each into thirds, then continued with an equation of the form 1/3 × 2 = 2/3, showing how a child could use an equation to represent an abstraction of their direct modeling solution.

The flexibility of teachers' knowledge was reflected primarily in this profile by the range of strategies anticipated within the strategy category of emergent anticipatory direct modeling. The majority of teachers' strategies in this group included two if not three different ways to partition and distribute pancakes using drawn models (into sixths, into thirds, and into halves and sixths). Interestingly, each teacher anticipated at most one anticipatory strategy and these tended to be transitional and contain elements of direct modeling strategies, such as the fourth strategy in Figure 2.

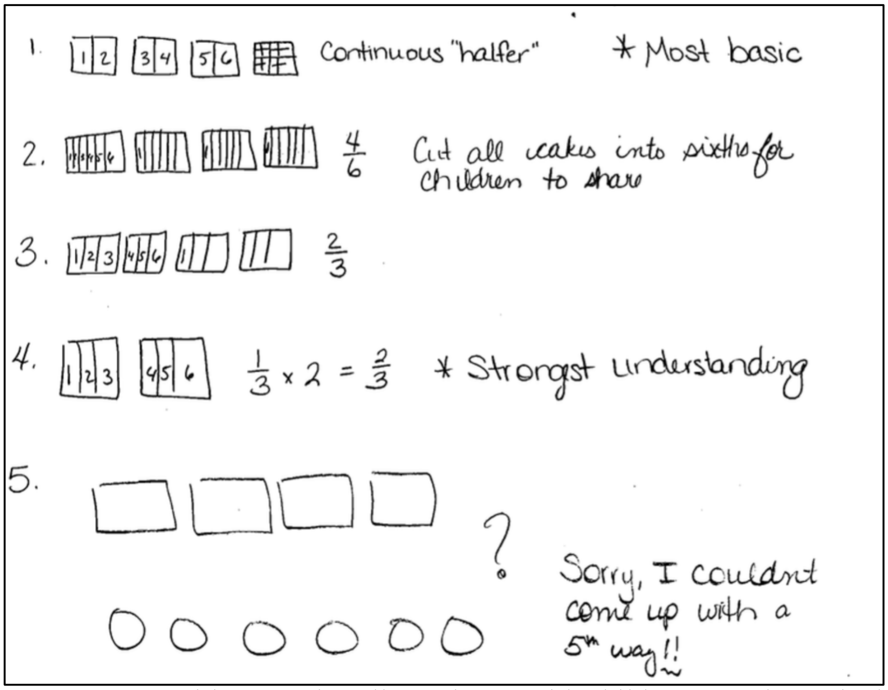

Figure 2. A response with strategies all consistent with children's mathematical thinking.

Profile 2: Limited evidence of knowledge of children's mathematical thinking

Teachers in this group anticipated a smaller number of children's typical strategies – one or two – and the majority of these strategies included details that were inconsistent with the way children might reason in the strategy. For example, virtually all of the direct modeling strategies in this group indicated that children would also simplify fractions to lowest terms or combine fractions with unlike denominators—actions that children who direct model are not likely to perform. The strategy on the left in Figure 3 provides an example that is consistent with what a child might do with a drawn model— the pancakes were cut into sixths and distributed one at a time—yet the simplification of the final answer of 4/6 to 2/3 is inconsistent with what a child using this strategy would likely understand about fractions. Similarly, another teacher anticipated that a child would draw a partition of the first three pancakes into fourths and distribute them, then draw the partition of the last pancake into sixths and distribute, and then combine the final amount, 1/4 + 1/4 + 1/6, numerically by finding a least common denominator of 12. A child who needed to draw the fractional quantities to make sense of the problem, as in these strategies, would likely not reason mentally to express fractions in lowest terms or combine using least common denominators.

The strategies anticipated by teachers in this group that were in line with children's typical strategies encompassed a smaller range than those anticipated in the first group. For example, teachers tended to anticipate only partitioning pancakes into sixths in their direct modeling strategies, and the rest of their valid anticipated strategies consisted of instructed procedures and conventions, such as long division or "invert and multiply" for fraction division (e.g., the set of strategies on the right in Figure 3). This combination suggests only a moderate amount of flexibility with respect to thinking like a child. Strategies in this profile were also not accurately assigned to least and most understanding. For example, one teacher indicated that an emergent anticipatory direct modeling strategy (involving sixths) represented the most understanding, whereas an anticipatory strategy that involved adding 1/6 four times represented the least understanding (when children's understanding of fractions actually develops in the opposite direction).

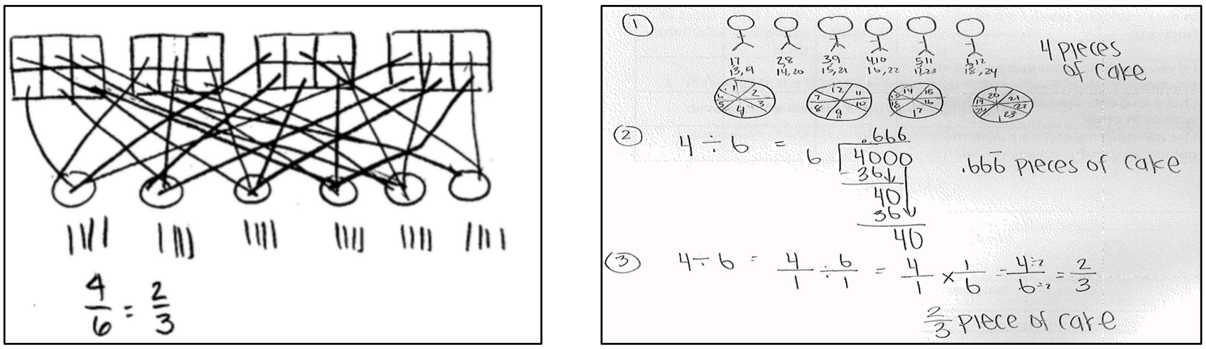

Figure 3. Responses with a mix of children's typical strategies and instructed procedures and conventions.

Profile 3: Lack of evidence of knowledge of children's mathematical thinking

The strategies anticipated by teachers in this group did not include any strategies that fully represented the reasoning children typically use in their strategies. Although each teacher thought that children would use drawn models to solve the problem, the details of how children would use these representations were inconsistent with how children would reason. For example, the strategy in Figure 4 on the left shows the pancakes partitioned into sixths, with the sixths distributed four at a time, an atypical process for a child relying on drawn models to solve this problem. Similarly, the strategy on the right shows the partitions distributed two at a time, also an atypical process for a child relaying on this drawn model. This distribution may suggests that the teacher solved the problem first and then used the answer as the basis for completing the drawn model, in contrast to distributing the pieces one by one, as a child who took the trouble to draw each pancake and partition it into sixths would likely do while solving the problem. Although these teachers had the intuition that children would use drawn models to solve this problem, there was a lack of evidence of knowledge of the typical informal strategies that children would use.

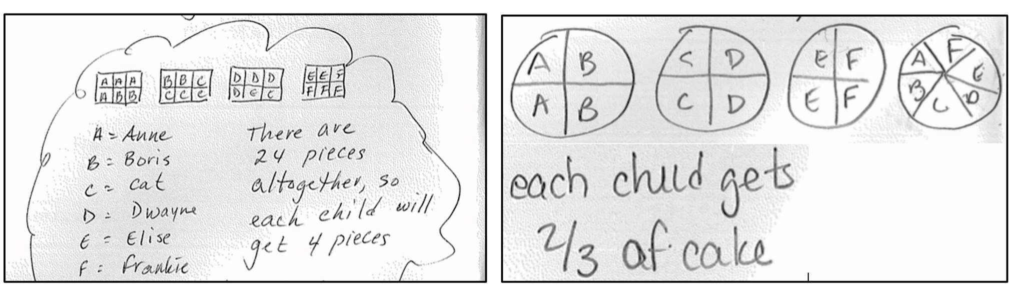

Figure 4. Strategies with details that are inconsistent with children's reasoning.

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Similarly to teachers in the second group, teachers in this group did not accurately assign strategies to least and most understanding. They also included some instructed procedures and conventions, though not as consistently as teachers in the middle group. Thus, teachers in this group appeared to have the least flexible knowledge with respect to "thinking like a child."

Conclusions

Imagining how a child would solve an equal sharing fraction problem such as the one in this study requires a teacher to see the problem through a child's eyes. It can be difficult to suspend one's own knowledge of mathematics to engage in thinking the way a child might. Yet teachers must do this to teach in ways that are responsive to children's thinking – they need *knowledge of children's mathematical thinking* to anticipate the strategies children might use, notice children's mathematical thinking, and respond appropriately with supporting and extending moves (Jacobs & Empson, 2016).

The majority of teachers in this exploratory study anticipated at least one strategy that was consistent with children's typical strategies as documented in past research. Among these anticipated strategies, direct modeling strategies were the most common. A handful of teachers anticipated nothing but direct modeling strategies or strategies using drawn models; and among those teachers who anticipated a small range of strategies consistent with children's typical strategies, a direct modeling strategy was always included. The pervasiveness of direct modeling in teachers' anticipated strategies suggests the accessibility of direct modeling as a way for teachers to make sense of children's mathematical thinking and to imagine how a child would solve equal sharing fraction problems.

Teachers' anticipation of direct modeling and other strategies allowed us to identify three knowledge profiles which varied along a continuum and were distinguished primarily by their flexibility in terms of thinking like a child. Flexibility with respect to the number of distinct and valid strategies provided a less useful measure by which to distinguish groups, although we conjecture that as a teacher's knowledge of children's mathematical thinking advances, there would be a corresponding increase in the number of valid and distinct strategies consistent with children's typical reasoning. We also conjecture that this number would include instructed procedures and conventions, although not at the expense of the variety of informal strategies children are known to use to make sense of an equal sharing fraction problem.

In summary we found the idea of flexibility as the extent to which teachers seemed to be able to think like a child not only useful for characterizing teachers' knowledge, but also an important extension to Whitacre's (2015) characterization of flexibility in terms of the number of distinct and valid strategies. The greater this flexibility, we conjecture, the greater teachers' ability to respond to children's ever evolving mathematical thinking during instruction. This exploratory study can be extended in two ways: investigate the knowledge of children's mathematical thinking in a larger sample of teachers systematically selected on the basis of their documented participation in professional development focused on children's mathematical thinking and use methods such as clinical interviewing that elicit teachers' reasoning for the strategies that they anticipate.

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