Teaching and Classroom Practice IMPACT OF A STUDENT-ADAPTIVE PEDAGOGY PD PROGRAM ON STUDENTS' MULTIPLICATIVE REASONING

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This study examines how a PD program to promote teachers' shift toward a student-adaptive pedagogy impacts students' multiplicative reasoning. We describe the underpinnings of this pedagogy and main components of the PD program. Then, we present key features of participants, data collection/analysis methods, and the written assessment used to measure students' multiplicative reasoning (MR). We found a significant increase in students' MR, between (a) year-ends for different classes and (b) year-start to year-end for the same groups of students. We also found students of participating ("treatment") teachers outperformed those of non-participating ("control") teachers. We discuss the importance of these findings for theory, for teacher education, and for students' mathematical future.

Introduction

In the context of NCTM (2000) reform efforts, we promote students' mathematics by fostering teachers' focus on students' reasoning. Here, we examine the impact of a professional development (PD) program to foster grade-3 teachers' shift to a constructivist-based, *student-adaptive pedagogy* (Tzur, 2013) on their students' multiplicative reasoning. We stress our focus is on students' reasoning, not on their observable solutions to problems (Tzur et al., 2013). In our study, we use students' work to infer reasoning in terms of mental operations on units that could explain a *child's underlying sense* of the problem and its solution. Our study contributes to the body of research that links teacher learning to identify and build on students' mathematics with students' learning and outcomes (Visnovska & Cobb, 2009).

Conceptual framework

We explicate two components of our conceptual framework: student-adaptive pedagogy and numerical reasoning—particularly the difference between additive and multiplicative reasoning. Extending Steffe's (1990) notion of adaptive teaching, Tzur (2008, 2013) proposed student-adaptive pedagogy as a comprehensive approach rooted in a conception-based perspective on knowing and learning. Simon et al. (2000) distinguished this perspective from traditional ("show-and-tell") and perception-based perspectives identified in teachers' transition to reform-oriented practices. Perception-based perspectives differ from traditional in emphasizing the active nature of learning mathematics. However, common to both perspectives is a stance that depicts mathematical knowing as existing outside the learner.

In contrast, a conception-based perspective builds on two implications of the core constructivist notion of assimilation (Piaget, 1985). First, one's available ways of operating afford and constrain what and how one may "see" and do mathematically. Second, conceptual learning entails bringing forth available ways of operating mathematically and transforming those into more advanced ones. These implications compel pedagogical practices that adapt goals and activities for students' learning based on students' available mathematics. Tzur (2008) depicted student-adaptive pedagogy as a reflective cycle (triad) of teaching rooted in hypothetical learning trajectories (HLT; see Simon, 1995; Simon & Tzur, 2004). It begins with inferring students' available mathematics, proceeds to setting the goals for their next learning, and then to selecting tasks that can be assimilated into available schemes and help transform those into the intended mathematics. For example, a teacher may infer two different strengths of her third graders' conception of whole numbers based on the strategy they use to add two single-digit numbers: weak (count-on) or strong (break-apart-make-ten, or BAMT) (Tzur et al., 2017). Thus, she would not set the same goals and engage all of them in the same tasks. Rather, she can engage those who used count-on in tasks to strengthen their conception of number and those who used BAMT in tasks to foster transition to multiplicative reasoning.

As for numerical reasoning, we explain it in terms of mental units and operations inferred to underlie students' problem solving (Steffe, 1992; Ulrich, 2016). Two types of units inform our inferences, *singletons (1s) and composite units* (i.e., units composed of smaller units). For example, the number "8" is a unit composed of eight 1s, or of five 1s and three 1s, etc. We infer *additive reasoning* when one's operations involve no unit change—she operates on one kind of unit (e.g., 2 *keys* + 2 *keys* + 2 *keys* = 6 *keys*). Conversely, we infer the first of six *multiplicative reasoning* schemes, which Tzur et al. (2013) termed *multiplicative double counting* (mDC), when one's operations involve a change of unit (Simon et al., in press). Such a change takes place when items of one kind of unit are distributed over (and coordinated with) items of another kind of unit to yield a different kind of unit (e.g., 2 *keys-per-box*, placed in 3 *boxes*, yield 6 *keys*). To find the total, a simultaneous count of accrual of composite units and 1s takes place (e.g., first-box-is-2-keys, second-is-4, third-is-6).

Methods

Settings and Participants

For two years, eleven grade-3 teachers and their students (age ~9) in two schools in a large USA city participated in the study. School A (7 teachers) is located in a small district and School B (4 teachers) in another, large district. In School A, 3 teachers participated in the PD program (treatment) and 4 teachers did not (control). Of the participating students, ~85% identified as students of colour and ~70% were learning English as an additional language (we detail student numbers in Data Collection).

Concept-Sensitive Assessment of the mDC Scheme

To assess students' mDC scheme, we used a 5-item written measure that our team developed and validated. Validation included correlating students' written responses with an interviewer's inferences of their mDC scheme (Ktb=0.883, $p<.0005$). The first item assesses additive reasoning; the following four assess the mDC scheme (Table 1). Cronbach's α (0.91) and Rasch (0.98) indicated inter-item reliability.

Table 1: Five Word Problems Comprising the mDC Written Assessment

#	Word Problem
1	For breakfast Ana ate 8 grapes. For lunch, Ana ate 7 grapes. How many grapes did Ana eat in all? Ana ate _______ grapes in all.
2	The picture shows towers made of **3** cubes, **5** cubes, **12** cubes, and **24** cubes. In this problem, Pat only has towers with **3 cubes**. Pat cannot break apart any tower. Can Pat build a tower of **24** cubes using only towers of **3 cubes**? (If Yes, fill in the blank): Pat CAN use _____ **towers** of **3** cubes to build a tower of **24** cubes.
3	The picture to the right shows a box. Alex put **6** towers in the box. Alex made each tower with **3** cubes. The numbers on the picture show this. How many **cubes in all** did Alex use to make **6** towers? (fill in the blank): ______
4	There are **4 teams** in a school. Each team has **5 players**. The picture shows the name of each team. Joy said there are **35 players in all**, because she skip-counted by 5 (Joy started 5, 10, and kept going to 35). Is Joy Correct? (circle one): Yes No. How many teams did Joy count? (circle one): 4 5 7 20 35 Another number ______
5	Sam baked **28** smiley **cookies**. He put all of them in boxes. Sam put **4 cookies in each box**. The picture shows only **one of the boxes**. How many boxes did Sam use for all 28 cookies? (fill in the blank): ________

Job-embedded PD Program

Focusing on multiplicative and fractional reasoning enacted within the teaching triad, we engaged participating teachers in a PD program to promote their: (a) own mathematical reasoning, (b) understanding of progressions in students' reasoning, (c) use of tasks to foster such progressions, and (d) attention to language and actions used by them and by students. To these ends, we used three, job-embedded PD experiences: Two, week-long *Summer Institutes* (Sis; total ~70 hours), *Buddy-Pairs* (total ~24 hours), and School/Grade-based workshops (total ~16 hours).

In both Sis, we engaged teachers in whole groups, small group, or individual work while using tasks they could later enact in their classrooms. We frequently involved them in observing conceptually-selected videos and analysing students' reasoning, In SI-2, the segments were selected from teachers' own classrooms. Using that analysis, they would discuss goals and tailor/justify tasks to promote the next learning.

During the school year (2016-17), we worked with teachers in their own classrooms, mingling *buddy-pair* experiences and grade-based workshops. In the former, teachers teamed up to visit a buddy's classroom, while a member of our team co-taught with the hosting teacher. Then, our team member and buddy teachers reflected on: (a) what students seemed to understand, (b) what serves as evidence for such claims, (c) how tasks could foster that learning, and (d) what/why/how to teach next. In the workshops, we focused on concepts from the buddy-pair experiences to promote teachers' own mathematics, to situate the concept(s) within progressions, and to select tasks that can promote differentiated learning based on where students seemed to be conceptually. A particular emphasis was on the strength of a student's conception of number and/or on the mDC scheme, which teachers learned to glean from students' strategies when solving problems in the classroom.

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Data Collection and Analysis

Collection

Five graduate research assistants (GRAs) administered the written assessment in a whole-class setting (~40 minutes). The GRA read out loud each item to enhance comprehension, monitored students' work on all sub-questions, and, before starting Problem 2, guided them to build a tower of 7 cubes to ascertain they recognized this object. We administered the mDC assessment three times: Spring '16 (pre-PD, year-end, N=81), Fall '16 (pre-PD, year-start, N=177), and Spring '17 (post-PD, N=113). The GRAs entered student responses to the mDC assessment in pairs; one read the responses out loud and another entered them into a spreadsheet.

Analysis

Scoring correct responses as 1 and incorrect as 0, we tested two main hypotheses about participating students: (a) post-PD, year-end outcomes will be better than pre-PD year-end and (b) post-PD, year-end outcomes will be better than pre-PD, year-start. We also tested a hypothesis that treatment students in School A will outperform their control counterparts. For each hypothesis, we analysed the mean of responses to all four mDC questions (ranging 0-4) and to Problem 3 alone (0 or 1). We chose Problem 3 because it is a typical multiplicative situation taught in schools that proved the hardest, and thus distinguishes teaching "multiplication as repeated addition" vs. as coordination of composite units and 1s. For the total mean on Problems 2-5 we used an independent sample t-test, a one-way ANOVA, a repeated-measure ANOVA, and Cohen's-d effect size (ES); for the non-parametric variable of responses to Problem 3 we used the Mann-Whitney test (MWz values).

Results

Three analyses show the PD impact on 3rd graders' reasoning: pre/post between two year-ends (Sp-16, Sp-17), pre/post growth between year-start and year-end (Fa-16, Sp-17), and pre/post between treatment and control students in School A.

Pre/Post PD: Year-Ends (Sp-16 vs. Sp-17)

Table 3 shows that, overall, 3rd graders at year-end of post-PD (39%) outperformed their year-end, pre-PD counterparts (28%), though this did *not* reach statistical significance (t=1.38, df=89, p=.17). For Problem #3, students in post-PD (43%) outperformed pre-PD (32%; MWz=1.02, N=91, p=.31). These results differ for each school. In School A (treatment only), a minimal change is indicated for all mDC problems, from pre-PD (25%) to post-PD (28%), with a larger (non-significant) difference on Problem #3 (22% to 36%, respectively). These non-significant results are highlighted differently when compared with changes in School A's control group. In School B, the pre-post PD increase on all four mDC problems (29% to 56%) was statistically significant (t=2.15, df=47, p=.037), but the increase on Problem #3 (36% to 52%) was not (MWz=1.07, N=49, p=.28).

Table 2: Percentages of students' correct solution (all mDC problems; Problem 3).

	All mDC Problems		mDC Problem **3**	
	Pre (Sp-16)	Post (Sp-17)	Pre (Sp-16)	Post (Sp-17)
All AdPed	28%	39%	32%	43%
School A	25%	28%	22%	36%
School B	29%	52%	36%	52%

Pre/Post PD: Year-Start (Fa-16) vs. Year-End (Sp-17)

Table 4 shows growth, from year-start (Fa-16) to year-end (Sp-17), for all students, then separately for each school. Growth in mDC reasoning, from year-start (14%) to year-end (39%), was statistically significant (t=4.91, df=147, p<.0005), with *large effect-size* (Cohen-d's ES=0.83). Similarly, results for Problem #3 show the growth from year-start (18%) to year-end (43%) was statistically significant (MWz= 3.36, N=149, p=.001), with *moderate effect-size* (Cohen-r's ES=0.58).

In School A, the growth on all four mDC problems, from pre-PD (mere 8%) to post-PD (28%), was statistically significant (t=2.96, df=62, p=.004), with *near-large effect-size* (Cohen-d's ES=0.75). Similarly, for Problem #3 the growth from year-start (6%) to year-end (36%) was statistically significant (MWz=2.87, N=64, p=.004), with *intermediate effect-size* (Cohen-r's ES=0.3). In School B, the growth on all four problems was remarkable, from pre-PD (18%) to post-PD (52%) (t=4.88, df=83, p<.0005), with a *very large effect-size* (Cohen-d's ES=1.15). Growth on Problem 3, from pre-PD (24%) to post-PD (52%), was statistically significant (MWz=2.5, N=85, p=.012), with a *moderate effect-size* (Cohen-r's ES=0.62).

Table 3: Percentages of students' correct solution (all mDC problems; Problem 3).

	All mDC Problems		mDC Problem **3**	
	Pre (Fa-16)	Post (Sp-17)	Pre (Fa-16)	Post (Sp-17)
All AdPed	14%	39%	18%	43%
School A	8%	28%	6%	36%
School B	18%	52%	24%	52%

Treatment vs. Comparison

Table 5 (Sp-16 vs. S-17) and Table 6 (Fa-16 vs. Sp-17) compare outcomes of 3rd graders' mDC reasoning, between treatment and comparison groups (School A only). In both tables, a two-way ANOVA (shown in the post-PD cells of the treatment group) is statistically significant in favour of the treatment group.

Table 4: Sp-16 vs. Sp-17 – percentages of treatment/control student correct solutions.

	All mDC Problems		mDC Problem **3**	
School A	Pre (Sp-16)	Post (Sp-17)	Pre (Sp-16)	Post (Sp-17)
Treatment	25%	28% ($F_{1,142}$=4.42, p<.037)	22%	36% (Friedman's $Q_{1,145}$=26.6, p<.0005)
Control	36%	20%	39%	15%

Table 5: Fa-16 vs. Sp-17 – percentages of treatment/control student correct solutions.

	All mDC Problems		mDC Problem **3**	
School A	Pre (Fa-16)	Post (Sp-17)	Pre (Fa-16)	Post (Sp-17)
Treatment	8%	28% ($F_{1,203}$=6.26, p=.013)	10%	36% (Friedman's $Q_{1,54}$=38, p<.0005)
Control	19%	20%	23%	22%

Summary of Analysis

We analysed third graders' responses to four, concept-sensitive items that, combined, indicate students' reasoning with the mDC scheme. We showed PD impact on that reasoning using three major comparisons: (a) pre-post increase between two consecutive *year-ends* within the same student population, (b) pre-post growth from *year-start to year-end* in the same schools within the PD year, and (c) interaction between outcomes of students of teachers in treatment and control groups within the same, single school. Those comparisons support our claim: PD to foster teachers' shift toward student-adaptive pedagogy (focus on multiplicative reasoning) can bring about desired growth in their students' multiplicative reasoning and problem solving.

A question arises of causes for between-school differences. While more data are needed, we note two plausible factors. First, teacher practices in each school reflected a different starting-point: mostly traditional at School A and mostly reform-oriented, perception-based perspective at school B (Simon et al., 2004). Second, teachers differed in their learning, and thus enactment, of assessing and using their students' conception of number, including how this conception predicts mDC (Tzur et al., 2017). Specifically, School B teachers reorganized instruction to (a) foster conception of number in students who seemed to lack it and (b) strengthen it in students with a weak conception of number. That is, they focused more on fostering students' construction of mDC by capitalizing on the strength of their conception of number.

Discussion

We found impact of a PD program to foster teachers' shift toward student-adaptive pedagogy on growth in their students' multiplicative reasoning. The scope of this paper precludes detailing the job-embedded PD. Yet, it shows that fostering teachers' initial adoption of this constructivist-based pedagogy supports the crucial conceptual advance in students' reasoning—from additive to multiplicative. Moreover, it shows the benefits of helping teachers to first conceptualize multiplication themselves, and then teach it, not as repeated addition (as did Control teachers), but rather as coordination of three different units, which we fostered in treatment teachers.

Implications for Practice

We note two main implications of this study for practice. First, our findings stress the benefits of changing teachers' understanding of multiplication, and then of teaching it, away from "repeated addition" and toward units coordination. Arguably, the most telling evidence is found in students' solutions to the *typical multiplication Problem 3*—markedly students of treatment teachers outperforming their control counterparts. Second, for mathematics teacher education, our study highlights the benefits of a dual focus on the necessary growth in teachers' own mathematical reasoning and their ability to tailor learning goals and activities to the students' available conceptions. Specifically, in this study we demonstrate the possibility of increasing students' multiplicative reasoning and problem solving by promoting teacher development as professionals who can understand, and apply, theory and research findings to alter their practice. We note that, in both schools, the principal and other instructional leaders provided constant, enthusiastic support for the intended teacher change.

Implications for Research and Theory

We note five main foci implied by this study. First, for theory building, this study implies the possibility, and need, to corroborate, statistically, the theoretically sound progression in students' schemes for whole number multiplicative reasoning (Tzur et al., 2013). In this study, we found the impact on students' mDC, an introductory level of multiplicative reasoning. Collecting and analysing data from a large sample of students would allow corroboration to more advanced schemes of multiplicative reasoning. Second, it seems important to also link the impact of our constructivist-based, job-embedded PD on students' mDC reasoning with their outcomes on district and/or state tests, as well as studying this impact for different student populations. Third, in our data analysis we grouped teachers uniformly, although they started their individual journeys toward student-adaptive pedagogy at different points and progressed in rather different paces. Linking differentiations in teacher practices to student learning and outcomes could further understandings of how the impact found in this study is related to teacher learning. Fourth, further attribution of PD impact to parts of the intervention is needed, that is, to changes in teachers': own mathematics, understanding of conceptual progressions in students' mathematics, selection and enactment of instructional activities tailored to fostering particular students' learning, and use of language as an additional focus of the intervention. Fifth, the differentiated support by school principals and coaches we witnessed indicates the importance of considering an extended-level unit of analysis (beyond teachers), namely, schools as systems. All five foci can build on the findings reported here about the promising impact that a constructivist-based PD can have on teachers and thus on students' mathematics.

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