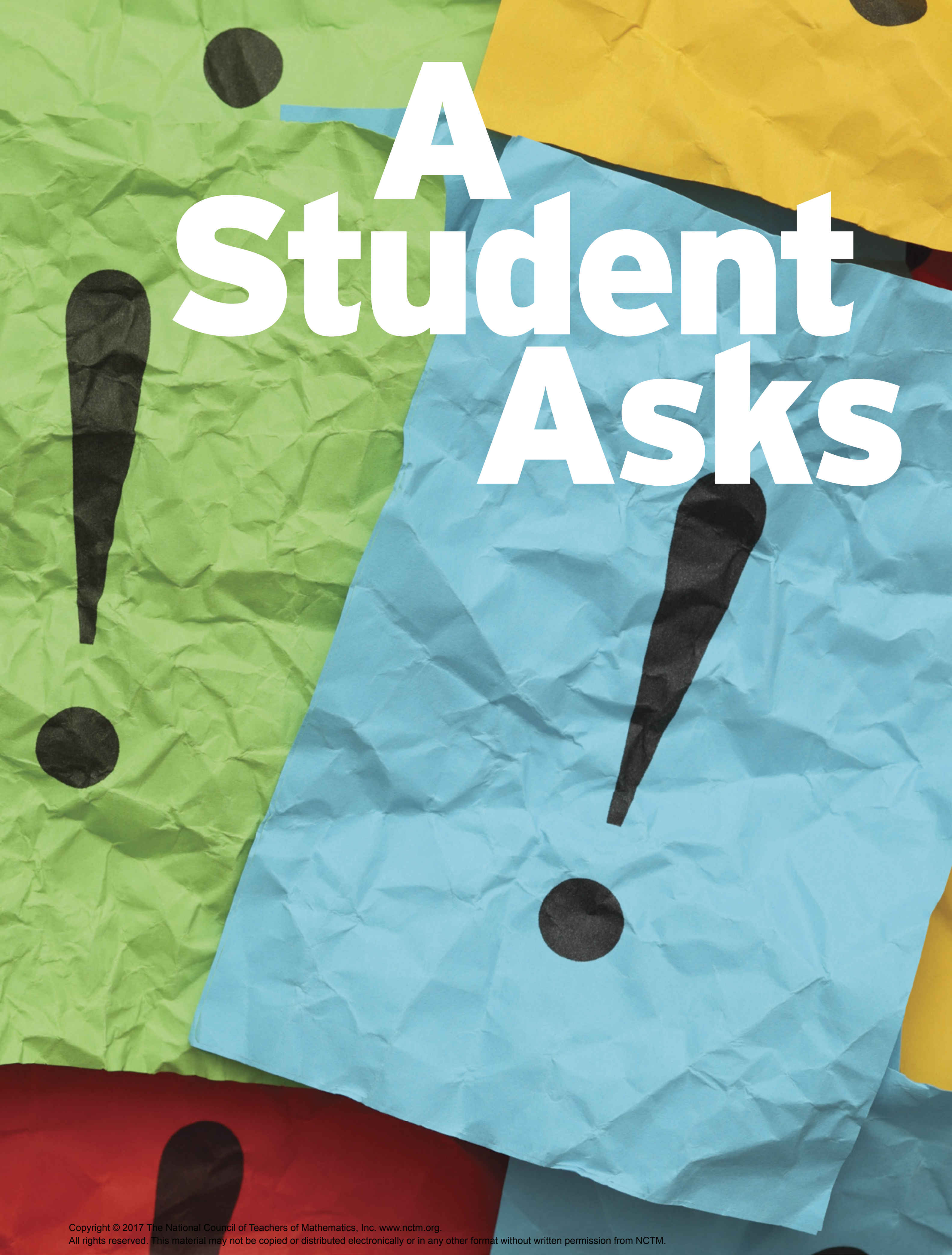
A Student Asks



A first-year algebra student's curiosity about factorials of negative numbers became a starting point for an extended discovery lesson into territory not usually explored in secondary school mathematics.

about (−5)!

E. Paul Goldenberg and Cynthia J. Carter

A first-year-algebra class was considering how many ways five people could be arranged in line. Working together, they did not take long to hatch the idea that any of the five could be first and, for each choice, any of the remaining four could be second: so far, 5×4 choices. For each of those possibilities, any of the remaining three people could go next: $5 \times 4 \times 3$. Two remain for fourth place: $5 \times 4 \times 3 \times 2$. Now there is "no choice" (or "one choice," depending on how you like to say it) about the last person. Many students calculated each step, arriving at 120 when "no choice" remained; few had explicitly written the $5 \times 4 \times 3 \times 2$ structure. When we reviewed the logic, the unfinished look of $5 \times 4 \times 3 \times 2$ moved some students to suggest $5 \times 4 \times 3 \times 2 \times 1$. The value does not change, but it looks more elegant. A few students recognized this structure and knew it was called "factorial," which they explained informally as "multiply all the numbers down to 1."

As class ended, Noa asked if negative numbers could have factorials. This content fits nowhere in precollege mathematics, but taking a students' mathematical curiosity seriously is good teaching.

Her question did fit a common theme of this class—how to make sense of mathematics and extend it into new territory. For the students, this mostly meant extending in a consistent way from positive integers to the real numbers. By thinking about extension, we could tie content to mathematical thinking as articulated in the Common Core's Standards for Mathematical Practice (SMP) and elsewhere (CCSSI 2010; Goldenberg et al. 2015). The students had most recently worked through the mathematical idea of "extension" to make sense of exponents in expressions like 5^{-2} and $9^{1/2}$.

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We approached extension by starting with what we called a "natural meaning" and then trying to extend it in a way that "remained useful" and "did not break anything." These informal, almost frivolous phrasings were deliberate, intended to convey that our goal was to *make sense of something new*: we were *creating* mathematical ideas, not attempting to discover (or look up) some inviolable law of nature that we just happened not to know yet. This is the stance that mathematicians take. The students understood well that extension was a *creative* act, not a formulaic one. It required curiosity, experimentation, judgment, logic, and a sense for elegance. They had already shown that sense for elegance in their preference for $5 \times 4 \times 3 \times 2 \times 1$ over $5 \times 4 \times 3 \times 2$. We want them to see that mathematics itself, not just its application, involves creativity and esthetics. Waiting for that until graduate school is too late; by then, we have lost too many creative mathematical thinkers.

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Defining zero factorial met a practical need, but this new romp into (−5)! started because of Noa's curiosity, a problem posed by a student, not by a book.

To extend exponentiation, students returned to the reason for that "shorthand" notation. Repeated multiplication

$$\underbrace{a \times a \times a \times \cdots \times a}_{n}$$

is useful enough to warrant a name (exponentiation) and shorter notation a^n. We do not invent simplified notations for calculations that we seldom need. In *Girls' Angle Bulletin*, Ken Fan (2012, p. 14) writes ". . . much notation . . . starts as a matter of convenience, and sometimes takes on a life of its own." But that "natural meaning" for exponentiation—repeated multiplication—makes sense only when n is a counting number greater than 1. "Four 3s multiplied together" (3^4) makes sense, but "one 3 multiplied together" is stretching it. Multiplied together with what? And "zero 3s multiplied together" is total nonsense. Here is where the notion of extension is needed.

We would *like* to be able to use the symbol 3^n without restrictions, so we try to invent sensible meanings, consistent with the natural meaning, for unnatural circumstances like 3^1 and 3^0. The *natural* meaning of $a^n \times a^m$ is

$$\underbrace{a \times a \times a \times \cdots \times a}_{n} \cdot \underbrace{a \times a \times a \times \cdots \times a}_{m},$$

which we summarize as $a^n \times a^m = a^{n+m}$. We can use that to make sense of the (so far) nonsensical a^0. For $a^0 \times a^m$ to equal a^{0+m} (which is just a^m), we *want* a^0 to be 1. This is not a "natural" meaning—we invented exponentiation as repeated multiplication, and a^0 is not that at all—but it is a consistent extension: it remains useful and does not break anything.

We can also define 2^{-3} by extension. By our *natural* meaning, $a^n \times a^m = a^{n+m}$, so we would want $2^{-3} \times 2^3$ to be 2^0. Our natural meaning for 2^3 is 8; we are now content that 2^0 is 1; so, to remain consistent, we *want* 2^{-3} to be 1/8. And we can affirm in other ways that this makes sense. Starting with 16, repeatedly divide by 2, getting 8, 4, 2, 1, ½, 1/4, 1/8, and so on. Representing that sequence with exponents, we get 2^4, 2^3, 2^2, 2^1, and 2^0, so the notation 2^{-1}, 2^{-2}, 2^{-3} continues to make sense: It remains useful and nothing breaks.

We can go further. Just as "zero 9s multiplied together" is nonsensical, there is no *natural* meaning for $9^{1/2}$. But, for a sensible extension, we would *want* $9^{1/2} \times 9^{1/2} = 9^1$ (which we happily agree is 9), so we *want* $9^{1/2}$ to be 3. (See EDC 2016a, 2016b; Mark et al. 2014; Goldenberg et al. 2015.)

Saying "this is what we *want*" rather than "this is what *must* be" may also seem frivolous, but it is important. Although our "wants" cannot be arbitrary—mathematicians do not just make stuff up—the way we construct definitions depends both on logical consistency and on what feels useful.

BACK TO NOA AND (−5)!

Just as multiplication and exponentiation get special names and symbols because they are useful, so does factorial. We do not give

$$5 \times (4½) \times 4 \times (3½) \times 3 \times (2½) \times \cdots$$

a special name and symbol because that computation is not useful enough. So, to extend the factorial, we need to see why we care about it at all and what we *want* its value to be in "unnatural" cases.

The class already knew about arrangements: 5 people can be arranged in $5 \times 4 \times 3 \times 2 \times 1$ ways. We use 5! as the shorthand for that computation.

The class also knew that to *choose* 3 people out of 5, they had 5 choices for "first" person, 4 choices for "second," and 3 choices for "third," or $5 \times 4 \times 3$ possibilities. (A confusion lurks here; we will deal with it later.) This *looks* like the beginning of a factorial calculation, so elegance suggested writing $5 \times 4 \times 3$ using factorials: 5!/2!. Ah, but we do not care who is “first,” “second,” or “third,” so we must divide this result by the number of ways of *arranging* the people we chose. There are 3! arrangements of three people, so we now have $5!/(2! \times 3!)$. We would be delighted if that calculation works for choosing 3 people out of 7, so we check the logic (and the actual counting) to see that it does. The “natural” meaning we gave to the factorial continues to serve well *until* we try to choose 1 person out of 5. We know the answer—5 choices—but, for elegance, we want our computation $5!/(1! \times 4!)$ to work. We have a *natural* meaning for 5! and 4!, but “multiply all the numbers down to 1” makes little sense for 1!. Our computation yields 5 only *if we claim* $1! = 1$. And what if we choose all 5 people out of 5? Again, we know the answer—“there’s one way; take ’em all”—but we would still like our formula to work. It says $5!/(5! \times 0!)$. The factorial’s *natural* definition makes no sense for 0!, but we can salvage the formula *if we claim* that $0! = 1$.

But *claiming* is not mathematics. We cannot claim that $2 + 2 = 7$ even if we want to. Let’s see if we can make our claims about the factorial feel less arbitrary. The *natural* meaning tells us that

$$\begin{aligned} 4! &= 4 \times 3 \times 2 \times 1 \\ &= 4 \times (3 \times 2 \times 1) \\ &= 4 \times 3!. \end{aligned}$$

Generalizing, $n! = n(n - 1)!$

Using $3! = 3 \times 2 \times 1$ as an example of a “natural” starting place, we will use $n! = n(n - 1)!$ as (the rest of) a *definition* to derive new cases from ones that we already have. We have already seen how to get 4! from it, and we can easily get the factorial of any larger integer n that way. This combination—a single case we know and a way of deriving new cases from it—is called a *recursive* definition. Rearranged, $n! = n(n - 1)!$ says that $(n - 1)! = n!/n$, which lets us give meaning to 1! and 0! in ways that “remain useful” (in other words, they satisfy our wants) and “don’t break anything” (they fit definitions of basic operations and behavior of the particular function—in this case, the factorial). Using the recursive definition, $(n - 1)! = n!/n$, we get $1! = 2!/2$. That gives the value 1 that we wanted. And $0! = 1!/1$ is another happy result. To find the value of Noa’s $(-5)!$, keep taking steps backward!

The next step says that $(-1)! = 0!/0$. Uh oh. . . . We cannot take that step. We cannot divide by 0. Are we doomed?

Maybe. But we will not give up yet. Making sense of a problem and persevering in solving it (SMP 1) involves, among other things, “chang[ing] course if necessary.” We cannot *directly* get to $(-1)!$, but maybe we can sneak up on it gradually by inventing a way to think about the factorial

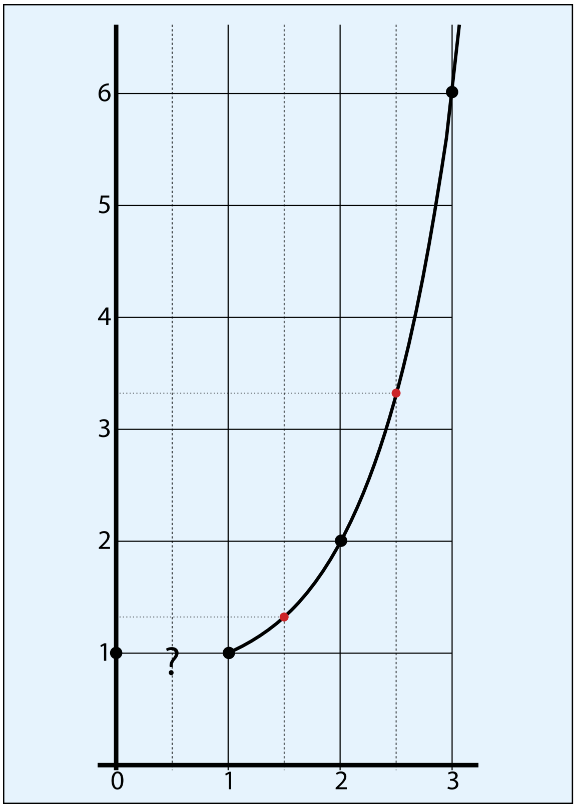

Fig. 1 A “smoothed out” factorial curve shows that $(2\frac{1}{2})! \approx 3.3$ and that $(1\frac{1}{2})! \approx 1.3$.

between integers, interpolating values between the natural ones. For example, what might we want $(2\frac{1}{2})!$ to be?

When we defined 1! and 0!, we knew we *wanted* them to equal 1, and we found a way to get that value while remaining consistent with the natural meaning of the factorial and its recursive description. But for $(2\frac{1}{2})!$, we have no starting place: neither a rock-solid idea of what value we *want* (as we had for 1! and 0!), nor such a rule as $a^n \times a^m = a^{n+m}$ that arises from natural meaning and allows us to interpret $a^n \times a^m$ even when n and m are not integers (as long as $n + m$ is an integer).

Taking students’ mathematical curiosity seriously is good teaching.

Intuition makes us *want* $(2\frac{1}{2})!$ to be between 2! and 3!, and closer to 2!, just as 2! is between 1! and 3!, and closer to 1!. Intuition does not always win, but it is a decent starting place. So, we plot 0!, 1!, 2!, and 3!, sketch as smooth a curve as we can through these points, and think. (See **fig. 1**.) Hmmm. . . . What happens between 0! and 1? Should the graph go straight across, staying 1 throughout that whole interval? Should it curve slightly below and then come back

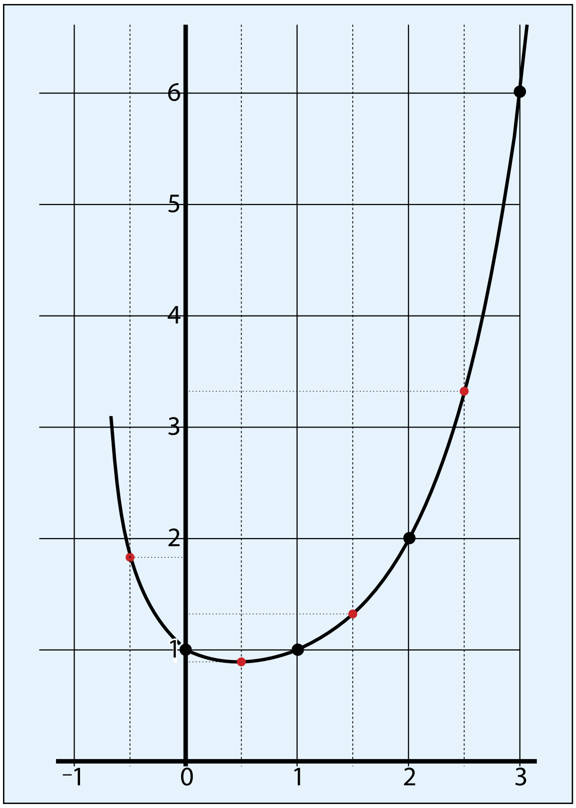

Fig. 2 Two new points, (1½)! ≈ 0.87 and (−½)! ≈ 1.77, allow us to sketch more of the curve.

up? For now, we will leave that blank and return to it later. We do not yet know what to expect.

To take this sketch seriously, we would *like* (2½)! to be about 3.3 and we'd *like* (1½)! to be about 1.3. Recall, we have no formula and no natural definition; we are just eyeballing numbers from a rough sketch. We are experimenting.

Confidence that we are on the right track would increase if the natural definition still applies and (1½)! = (2½)!/(2½). Amazingly, even with our rough sketch and estimates, 1.3 ≈ 3.3/2.5. Wow! "Making assumptions and approximations to simplify a complicated situation" (SMP 4) really *is* a powerful way to approach problems.

What about (½)! = (1½)!/(1½)? The computation (½)! = (1.3)/(1½) ≈ 0.87 lets us take yet another step, (−½)! = (½)!/(½) ≈ 1.77.

Wow again! This looks quite well behaved. So far, at least, it still fits our intuition, and we can keep going, getting more positive and negative half-integer values (see **fig. 2**).

We took this approach to see if we could sneak up on (−1)!, which we could not reach directly. Students could approximate (−0.9)! the same way they approximated (−½)!, by estimating, say, 2.1!. It looks roughly like 2.2. Working backward, the definition let them compute 1.1!, then 0.1!, then (−0.9)! ≈ 9.5. And they could eyeball 2.9! ≈ 5.3, divide by 2.9 to get 1.9! ≈ 1.8, divide by 1.9 to get 0.9! ≈ 0.96, divide by 0.9 to get (−0.1)! ≈ 1, and divide by −0.1 to get (−1.1)! ≈ −10. Wow yet again! In the tiny distance from −0.9 to −1.1, the value of the factorial somehow goes from high positive (~9.5) to low negative (~ −10). No wonder (−1)! has no sensible value.

To calculate factorials at half integers starting with (2½)!, as we described here, students first used an eyeball approximation (3.3), but we later supplied 3.32335—a more precise starting value than they could derive with their current mathematical background—which let them compute the value of (−½)! with enough precision to reveal a surprise.

Start with (2½)! ≈ 3.32335 and compute the value of (−½)! yourself, just as they did, and then square that result. Mathematics is full of surprises! That sentence *deserves* an exclamation point. First-year algebra students do not yet have the mathematical experience to *explain* that surprise, but they enjoyed the feeling of awe it brought them and felt proud.

To fill in fine details, students then graphed x! with Desmos®, https://www.desmos.com/calculator. (See **fig. 3**.)

Pedagogically, the timing matters. If students had used this tool at the outset, all it would have done is reveal an "answer": Factorials of negatives exist, and they look strange. Done. After the students' work, the tool supported their reasoning and led to new ideas and insights. The graph is weird, but the alternating sign of (−3½)!, (−2½)!, (−1½)!, (−½)! rings a bell. If $n! = n(n - 1)!$, then

Student questions, like this one, are not tame. What we can do immediately is say, "Wow, I need to think about that." That tells students that having to think is OK, and that there are things that you, as the teacher, do not know. It also tells students that you value their question enough to invest time in it.

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$(-\tfrac{1}{2})! = (-\tfrac{1}{2}) \times (-1\tfrac{1}{2})!$, and then $(-1\tfrac{1}{2})! = (-1\tfrac{1}{2}) \times (-2\tfrac{1}{2})!$, and so on. Of course the sign alternates! Unpacking, we would see that

$$
\begin{aligned}
&(-\tfrac{1}{2})! \\
&\quad = (-\tfrac{1}{2})(-1\tfrac{1}{2})(-2\tfrac{1}{2})(-3\tfrac{1}{2})(-4\tfrac{1}{2})!,
\end{aligned}
$$

something that looks a bit like the "natural" idea of the factorial with which we started. If we had an approximate value for $(-4\tfrac{1}{2})!$, we could approximate $(-\tfrac{1}{2})!$ on a plain calculator.

THINGS WE *DID NOT* DO TO EXTEND FACTORIAL

We did not make stuff up.

We did not, for example, arbitrarily decide that $(-5)!$ should be $(-5)(-4)(-3)(-2)(-1)$. That looks temptingly like our natural definition, but it does not pass the "doesn't break anything" test. It breaks the natural definition: multiplying n times $(n - 1)!$ should produce $n!$.

We also did not invent something like $3\tfrac{1}{2}! = 3\tfrac{1}{2} \cdot 2\tfrac{1}{2} \cdot 1\tfrac{1}{2}$. That preserves the $n! = n(n - 1)!$ rule but arbitrarily chooses where to stop. Why not $3\tfrac{1}{2} \cdot 2\tfrac{1}{2} \cdot 1\tfrac{1}{2} \cdot \tfrac{1}{2}$? Moreover, if $3\tfrac{1}{2}! = 3\tfrac{1}{2} \cdot 2\tfrac{1}{2} \cdot 1\tfrac{1}{2}$, then would $2\tfrac{1}{2}! = 2\tfrac{1}{2} \cdot 1\tfrac{1}{2}$, and $1\tfrac{1}{2}! = 1\tfrac{1}{2}$? Graph these made-up computations alongside the natural points for $n!$. They don't fit well. Although secondary school students do not know what use there might be for factorials of anything other than nonnegative integers, if there *is* a use, this awkwardly fitting made-up computation would seem unlikely to "remain useful."

THINGS WE DID DO

Defining 0! met a practical need, but this new romp into $(-\tfrac{1}{2})!$ started because of Noa's curiosity, a problem posed by a student, not by a book! Responding to curiosity fosters curiosity. But not all responses are equal.

Telling Noa, "If you major in mathematics in college, you will learn about the gamma function, and that will answer your question," is *not* rewarding. Nor is the answer, "Yes, negative nonintegers have factorials." Both defer further thought.

Responding well does not require that we teachers know all the answers. Our challenge is to find something that *fits current teaching goals and student capabilities* and that lets students use and hone their skills to address their own question. Sometimes we will be ready immediately, but more often we will be stuck needing to think harder than is possible while we are in the spotlight, or even needing to look things up. Student questions, like this one, are not tame. What we *can* do immediately is say, "Wow, I need to think about that." That tells students that having to think is OK, and that there

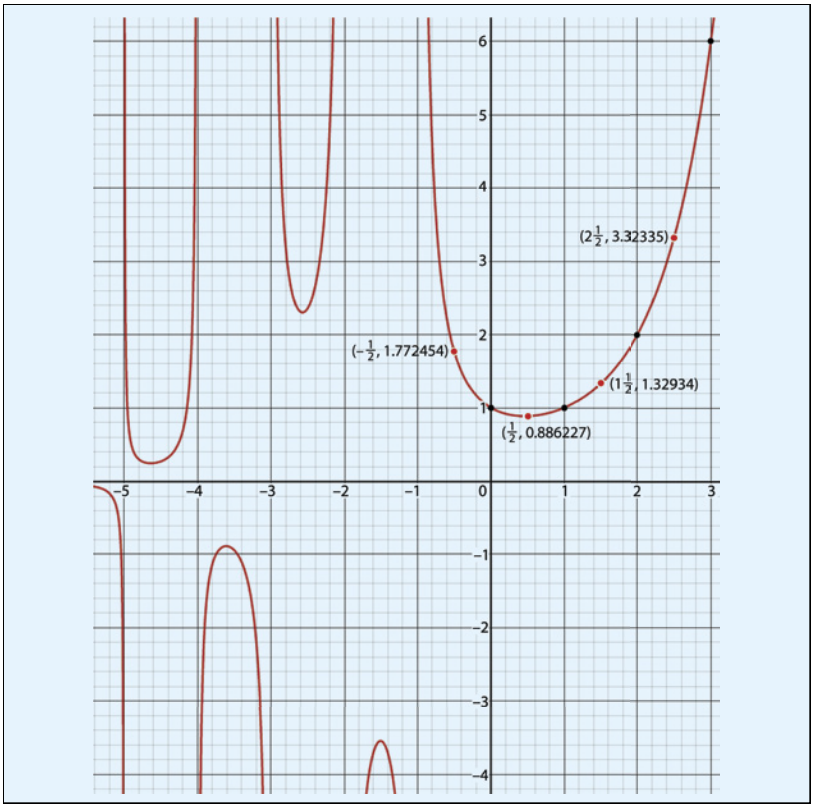

Fig. 3 The graph of *x*! has some values plotted.

are things that you, as the teacher, do not know. It also tells students that you value their question enough to invest time in it. It's also often the honest move, even if you know the mathematics well. Unexpected questions like Noa's almost always require us to take time to figure out how to make what we know (or learn) accessible to the students.

The methods that mathematicians historically used to derive a function that "smooths out" factorial are beyond first-year algebra, but the adventure described above is not: Noa's class found the challenge fun and could engage in the entire storyline and logic. With patience, a simple calculator, and a teamwork divide-and-conquer strategy, the students gathered enough results to make a fair approximation of a graph of the factorial on positive and negative numbers (excluding negative integers, which they could logically rule out). Being able to find a way to answer their own question was enormously motivating.

Finally, taking students' curiosity seriously shows students that *posing* problems—a genuinely new question not just a familiar problem with its nouns and numbers changed—is valued. It *should* be valued because it is valuable, partly as a super tool for problem solving and partly because it is what drives invention and discovery. Curiosity moves professionals, too. Some mathematical results that were only later recognized as useful were initially fueled partly by curiosity.

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MAKING SPACE FOR CREATIVE MATH

Mathematicians experiment, follow hunches and curiosity, and explore new ideas and territories. Too often, students (before graduate school) get a different image of math: formulas and tricks to be deployed.

The cost is high. Students who could be creative in mathematics, in computer science, or in other STEM fields abandon mathematics before they have a chance to discover its genuine appeal; many who do pursue higher-level courses carry an image of mathematics that does not include curiosity, creativity, and sense making.

Noa's class is different. Even young students can learn to ask real mathematical questions and figure out how to think through those questions, if questioning is encouraged and the questions are honored. But student questions are predictably not tame and can venture into places that are not in the curriculum. Making space—even a little space—to take the questions seriously helps the students take themselves seriously as mathematicians. Not all Noa's classmates will be mathematicians, but they all grew as mathematical thinkers and know the fun, challenge, and genuine surprises of *doing* mathematics, learning not only its rich body of knowledge but also what mathematical thinking is all about.

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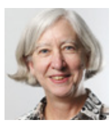

E. Paul Goldenberg, pgoldenberg@edc.org, has taught mathematics at all levels, early elementary through college, and now develops mathematics curriculum at EDC, a nonprofit organization in Waltham, Massachusetts. He especially likes problem posing that leads to depth, mathematical connections, surprise, and new ideas, and fostering students' questions. **Cynthia J. Carter,** ccarter@rashi.org, teaches mathematics in sixth, seventh, and eighth grades at The Rashi School, in Dedham, Massachusetts. She especially loves helping students *think* like mathematicians, posing problems as well as solving them, and building sense and system with their knowledge.

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