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This paper reports on a review of 12 empirical studies framed to address the problem of whether and in what ways mathematical knowledge for teaching influences teaching practice. From a larger review of literature on mathematical knowledge for teaching, this qualitative synthesis examines the theoretical foundations, methods applied and claims made. Most of the studies reviewed are small-scale qualitative studies. There is variability in the language to describe teaching and in how focused the studies are on teaching. We suggest that the tendency has been to focus on the question of the knowledge that teachers need, but that it would be more useful to focus on the mathematical entailments of doing *teaching, which will require more detailed and shared conceptualizations of the mathematical work of teaching.*

Keywords: Mathematical knowledge for teaching, teaching practice, literature review.

Introduction

In a review of literature on mathematical knowledge for teaching from 2006 to 2013, with colleagues we identify a number of studies that argue that mathematical knowledge for teaching influences teaching practice (Hoover, Mosvold, Ball, & Lai, 2016). The connection between mathematical knowledge for teaching and the quality of instruction is, however, complex. Hill, Umland, Litke, and Kapitula (2012) report evidence that weak mathematical knowledge for teaching predicts low instructional quality, and strong mathematical knowledge for teaching predicts high instructional quality, yet they also report that there is much more variation in teaching quality as well as in student achievement with teachers who perform in the midrange on measures of mathematical knowledge for teaching. Likewise, Hill et al. (2008) suggest that professional development, supplemental curriculum materials and teacher beliefs are all factors of potential influence, but these factors may cut both ways depending on the teachers' mathematical knowledge for teaching. In addition, efforts to clarify the conceptualization of mathematical knowledge for teaching continue to be concerned with the dynamic nature of mathematical knowledge for teaching, the usefulness of knowledge, and whether, when, and how it plays in teaching (Ball, 2016; Kersting, Givvin, Thompson, Santagata, & Stigler, 2012). From these different lines of work, it seems clear that decisions about teacher education or policy cannot be made simply by establishing *that* mathematical knowledge for teaching correlates with teaching practice. Although these correlational studies are arguably important, in this review we draw attention to the need to refine our understanding of *how* mathematical knowledge for teaching influences teaching practice.

The literature review

The present paper draws on results from a larger literature review of empirical research on mathematical knowledge that is specific to teaching (Hoover, Mosvold, Ball, & Lai, 2016). That review included a total of 190 articles that were coded for the following categories: 1) genre of study, 2) research problem used to motivate the study, 3) variables used, 4) whether or not and how causality was addressed, 5) findings. A research problem is an issue, topic, or question that motivates a study, indicating why the results would be of interest and how an investigation is linked to the literature. In most instances, the research problem was the same as what was specifically investigated, but at times there were tensions between the research problem and the research questions or the specific focus of the analysis. Distinguishing between research problems and the genre of the study helped us understand what a paper argues and how. In coding the research problem, we paid specific attention to the introduction and conclusion as opposed to the statement of the research questions or the specifics of the research design. In general, differences between the research problem and the research genre or design reflected inevitable tensions in the interrelated components of empirical research publications and provided useful insight into the approaches used in the study and the arguments made in the article. When considering the research problems that motivated the studies in our larger review, 12 studies focused on the ways in which teachers' knowledge contributes to practice. In this paper, we analyze these 12 studies with a particular focus on the theoretical frameworks of the studies, the methods applied and the claims made.

Studies of the influence of mathematical knowledge for teaching on teaching

The 12 studies framed to address the problem of whether and in what ways mathematical knowledge for teaching influences teaching practice have different characteristics. Seven studies investigate effects of mathematical knowledge for teaching on teaching practice (absent a specific intervention), one is an intervention study and four studies investigate mathematical knowledge for teaching as a construct in relation to teaching. Only one of the studies is quantitative, whereas most studies are small-scale qualitative studies. The participant teachers in the studies teach mathematics in primary, middle, and secondary schools, as well as at the university level — most of them are practicing or experienced teachers (see table 1).

Study	Sample size	Type	Causal design	Experience and level of teachers	Region	Teaching studied
Bansilal (2012)	Small (n=1)	Effect	Qualitative	Practicing secondary	Africa	Identifying key ideas, organizing explanations, listening to students
Cengiz et al. (2011)	Small (n=6)	Effect	Qualitative	Experienced primary	North America	Extending student thinking
Charalambous (2010)	Small (n=2)	Effect	Qualitative	Practicing primary	North America	Using representations, giving explanations, Interpreting and responding to student thinking
Choppin (2011)	Small (n=1)	Nature	None	Experienced middle school	North America	Engaging students with challenging tasks
Izsák et al. (2008)	Small (n=1)	Effect	Qualitative	Practicing middle school	North America	Using number lines for fraction addition
Johnson & Larsen (2012)	Small (n=1)	Effect	Qualitative	Practicing tertiary	North America	Listening to student thinking

Nardi et al. (2012)	Medium (n=11)	Nature	None	Practicing secondary	Europe	Identifying task objectives, interpreting and responding to student thinking
Rowland (2008)	Medium (n=24)	Nature	Qualitative	Future primary	Europe	Selecting and using examples
Speer & Wagner (2009)	Small (n=1)	Nature	Qualitative	Practicing tertiary	North America	Social and analytic scaffolding
Steele & Rogers (2012)	Small (n=2)	Effect	Qualitative	Practicing secondary	North America	Integrating different ideas of proof and positioning students as observers, creators, and explainers
Sullivan et al. (2009)	Large (n=97)	Intervention	Statistical	Practicing all levels	Oceania	Converting tasks to lessons
Tanase (2011)	Small (n=4)	Effect	Qualitative	Practicing primary	Europe	Connecting place value to other mathematical concepts, setting objectives, challenging students

Table 1: Studies investigating influences of mathematical knowledge for teaching on teaching

Next, we describe these studies with a focus on what they investigate, their methods, how teaching is conceptualized, and what we can learn from them. Cengiz, Kline and Grant (2011) focus on how teachers' MKT supports their teaching. They develop an extending-student-thinking framework based on analysis of instructional actions within episodes. In their investigation of six experienced elementary teachers, they draw upon Ball et al.'s (2008) conceptualization of mathematical knowledge for teaching. It is assumed that the participating teachers, due to their experience, have well-developed MKT. From analysis of video-recorded classroom observations and teacher interviews, these researchers provide detailed accounts of teaching and "demonstrate that MKT matters in the way teachers pursue student thinking" (Cengiz et al., 2011, p. 372). Their analysis of data from one of the participating teachers "provide evidence that a lack of certain aspects of knowledge can negatively impact a teacher's pursuit of student thinking" (p. 372). Similarly, Izsák, Tillema and Tunç-Pekkan (2008) provide fine-grained details in their analysis of the cognitive structures used by a teacher and her students when using number lines as a representation for fraction addition. Audio- and video-recorded interactions of a practicing middle-school teacher and her students formed a starting point for interviews with three pairs of students. Excerpts from lesson and student interviews were then used in a video-elicited interview with the teacher. They argue that subtle variations in the teacher's approach to partitioning unit intervals matter for the students' opportunities to learn.

Several studies are situated in the teaching of particular mathematical content. Steele and Rogers (2012) investigate the relationship between mathematical knowledge for teaching proof and teaching practice by combining clinical assessments with classroom observations of two secondary teachers — a novice and an expert teacher. Data collection included lesson observations, pre- and post-lesson interviews, written assessments and semi-structured interviews after the observation. The authors argue that the more experienced and MKT-knowledgeable teacher not only enacts a stronger and more nuanced lesson on mathematical proof, but her students end up having more mathematical authority. They argue that their use of MKT as a frame for examining practice provides an innovative method for investigating both MKT and features of instruction, such as student positioning as a key, mediating factor between MKT and opportunities to learn.

A study by Tanase (2011) investigates the connection between four Romanian first grade teachers' mathematical knowledge for teaching place value and their classroom practice. The participants are selected from a well performing and an average performing school in Romania. One experienced and one less experienced teacher from each school is selected for participation, and data collection includes teacher interviews, classroom observations and student assessments. Although all four teachers display good understanding of place value, Tanase suggests that teachers' knowledge goes beyond their own mathematical understanding. Differences are observed in teachers' ability to make connections between place value concepts and other mathematical concepts, how they set different objectives for students as well as the extent to which they challenge students in their mathematical work. Tanase also observes that although teachers have strong mathematical knowledge for teaching, and this knowledge impacts the quality of their instruction, their students may still not perform well. She suggests that student achievement is also influenced by external factors inside and outside of school.

Among these initial investigations of the specific influence of mathematical knowledge for teaching on teaching, most studies focus on teachers in primary, middle or secondary school. Two studies focus on mathematics teaching at university level. Speer and Wagner (2009) examine one undergraduate instructor's use of constructs of social and analytic scaffolding as a frame, the authors argue that aspects of pedagogical content knowledge are important for helping students find productive ways of solving particular problems and for understanding which student contributions, whether correct or incorrect, are important to emphasize in a discussion. They trace ways in which teachers' particular knowledge of students' understanding aids them in assuring that the lesson reaches intended mathematical goals and in understanding the role of particular mathematical ideas in students' development.

Another example is Johnson and Larsen's (2012) study of how a university teacher's mathematical knowledge influences her ability to listen when teaching abstract algebra. Their investigation focuses on how this particular aspect of mathematical knowledge for teaching supports mathematics teachers' listening when implementing a reform curriculum. Their theoretical framework distinguishes among three types of teacher listening: hermeneutic, interpretative and evaluative. Drawing on Speer and Wagner's (2009) argument that teacher listening requires particular types of mathematical knowledge for teaching, Johnson and Larsen examine the role of knowledge of content and of students in hearing tertiary students as they engage in reinventing the group concept in abstract algebra. Based on analyses of three teachers' classroom interactions when implementing a particular reform curriculum, Johnson and Larsen report on a teacher whose classroom interactions contained several episodes where the students were confused and the teacher was unable to make sense of their struggles. They observe that this teacher's ability to listen to her students draws on her knowledge of content and students. Johnson and Larsen posit that teachers need not only knowledge of students' misconceptions, but also knowledge of when and why students are likely to be confused and display misconceptions and of the consequences of such misconceptions when students engage in new activities.

The focus on teacher listening is also prevalent in Bansilal’s (2012) investigation of how a South-African mathematics teacher’s poor mathematical knowledge influences her classroom interactions. In this case study, the focus is on the process-object understanding of ratio. Based on narrative analysis of field notes and transcripts from five lesson observations with interviews, Bansilal organizes her claims around three emerging themes. First, she argues that the teacher displays limited understanding of ratio in her teaching. Second, she argues that the teacher fails to identify key ideas and organize her explanations in a way that enables the students to notice the big ideas involved in the mathematical task. Third, Bansilal points to the stressful environment that the teacher experiences in this classroom and suggests that this environment is caused by her lack of knowledge of the students as well as her preference for evaluative rather than interpretative listening.

In his study of mathematics teacher knowledge and its impact on how teachers engage students with challenging tasks, Choppin (2011) explores pedagogical content knowledge as situated in an instructional sequence. From his study, he aims at exploring teachers’ “local theory of instruction”. Choppin investigates an experienced middle-school mathematics teacher while she is teaching a particular curriculum unit over two years. In order to investigate the teacher’s knowledge, interview data are analyzed with a focus on her articulation of “(1) how student thinking develops over time, (2) the process by which that thinking develops, and (3) the resources that facilitate the development of student thinking” (p. 12). Based on his analysis of data, Choppin claims that the teacher develops her local theory of instruction from teaching. The teacher’s knowledge appears to influence her teaching in several ways, for instance in her adaptation of tasks.

Engaging students with challenging tasks is an important component of the work of teaching mathematics, and so is the selection and use of appropriate examples. Rowland (2008) focuses on mathematics teachers’ purposes for using examples in elementary mathematics teaching. Video recordings from 24 lessons taught by 12 pre-service elementary teachers are analyzed from a grounded approach, and codes are developed that focus on aspects of their teaching practice. The resulting 18 codes — one of the most common codes is “choice of examples” — are then placed in four overarching categories that constitute Rowland’s conceptualization of mathematical knowledge in teaching, commonly referred to as the knowledge quartet.

Although eight of the studies reviewed investigate effects of mathematical knowledge for teaching on mathematics teachers’ classroom practice, only one applies standardized measures of mathematics teacher knowledge. In his exploratory study, Charalambous (2010) investigates the connection between two primary teachers’ mathematical knowledge for teaching and their use of mathematical tasks. The two primary mathematics teachers had different levels of mathematical knowledge for teaching — as measured by MKT measures — and notable differences were found in how they planned, presented and implemented mathematical tasks. Charalambous applies Stein and colleagues’ mathematical tasks framework to examine the cognitive level of enacted tasks, and he formulates three tentative hypotheses about mechanisms of how mathematical knowledge for teaching impacts teachers’ selection and use of mathematical tasks. First, he hypothesizes that strong mathematical knowledge for teaching may contribute to a use of representations that supports students in solving problems, whereas weaker mathematical knowledge for teaching may limit instruction to memorizing rules. Second, he proposes that mathematical knowledge for teaching appears to support teachers’ ability to provide explanations that give meaning to mathematical procedures. Third, he proposes that teachers' mathematical knowledge for teaching may be related to their ability to follow students' thinking and responsively support development of understanding.

The study of Nardi, Biza and Zachariades (2012) differs from many of the other studies on how teachers' knowledge influences their teaching practice in that they do not study observed teaching. Instead, these researchers analyze teachers' argumentation about hypothetical classroom scenarios in task-based interviews. From their analysis of eleven teachers, they suggest that the teachers' warrants for the claims made about these classroom scenarios are not always mathematical. Their argument, which has potentially interesting methodological implications, is that analysis of the argumentation provided by teachers in such task-based interviews may provide insight into how the teachers' knowledge and beliefs influence their classroom interactions.

Sullivan, Clarke and Clarke (2009) also investigate the influence of teacher knowledge on the planning phase of teaching. In particular, they investigate the assumption that teachers are able to convert tasks to lessons easily. From their analysis of 107 primary and secondary teachers' responses to questionnaire items — and interpreting the responses by using the subcategories of MKT — they observe that many teachers find it difficult to translate tasks to lessons. For instance, many teachers find it difficult to convert the task of determining which of $\frac{2}{3}$ and $\frac{201}{301}$ is larger into a worthwhile learning experience for students.

Discussion

With regard to research design and choice of methods, we observe that most of the studies are small-scale qualitative studies that explore the connections between mathematics teacher knowledge and teaching practice in different ways. Although many studies draw on a similar conceptualization of mathematical knowledge for teaching, only one study applies one of the existing standardized measures of such knowledge (Charalambous, 2010). Several studies present innovative methods to investigate contributions of teachers' mathematical knowledge to teaching practice, such as video-elicited interviews and hypothetical classroom scenarios in interview prompts. As we have argued elsewhere (Hoover et al., 2016), given that research is this arena is in early development and to date we lack clear, replicable methods, scholars' efforts to innovate seem well placed. Ideas proposed in these dozen papers contribute to that development.

Each of the 12 studies reported is concerned with uncovering what, how, and why mathematical knowledge for teaching matters for teaching, yet the overall picture is unclear. One issue may be that an effort to show *that* mathematical knowledge for teaching matters (a focus on impact) may lead to holding knowledge and teaching at arms length in ways that obscure the dynamic nature of the role of that knowledge in teaching. For instance, several papers argued that teachers' lack of knowledge constrained what they were able to see, hear, and do, without taking the additional step of elaborating what knowledge arises in the work, when, where, and how. We suggest that the field would profit from studies that examine the interplay between knowledge and teaching practice and that impact studies are better conducted at a larger scale once clear conceptual and measurement tools are in place. Another issue may be that the conceptualization of and focus on *teaching* in these studies is underdeveloped. Some of the studies examine what might be better described as features of instruction than as teaching practice. For instance, Steele and Rogers (2012) examine the degree to which different ideas of proof are integrated into instruction and how students are positioned in relation to mathematical explanation. We agree that these are important, but would like to understand more fully what it is that teachers need to do to integrate ideas and position students and what the mathematical entailments are for doing so. Some of the studies address constrained, specific tasks of teaching (cf. Hoover, Mosvold, & Fauskanger, 2014), such as selecting and using examples, while others are broad and general, such as engaging students with challenging tasks. What is meant by "teaching" and its role in these studies vary.

Progress on the problem of whether and in what ways mathematical knowledge for teaching influences teaching practice will require building more shared language for talking about teaching, starting with more explicit attention to how it is conceptualized and continuing through the development of more widely shared conceptualizations of the work of teaching. It will require more focused examination of what it takes to do teaching, conceptualized as meaningful work, supportive of learning and doing the work in professional community. As we have argued elsewhere (Hoover et al., 2016), this may need to go hand in hand with developing the theoretical foundations of research on teaching. Teaching is a professional practice engaged in human improvement work. While there are other important aims of education, teaching is centrally about supporting the learning of subject matter. Understanding the theoretical implications of these observations and acting on them may strengthen research and practice.

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