Math Teachers' Circles: Partnerships between Mathematicians and Teachers

B. Donaldson, M. Nakamaye, K. Umland, and D. White

Mathematicians depend on primary and secondary teachers to educate young students and inspire their mathematical minds. Teachers in turn benefit from mathematical experiences led by mathematicians that kindle their creative energy and broaden their intellectual horizons. Mathematics departments, as part of their mission, provide mathematical training for current and future teachers. Many mathematics faculty have a natural desire to share their excitement for mathematical inquiry in ways that can reach beyond their own classrooms and affect K–12 education. Unfortunately, mathematicians who are looking for opportunities to interact with teachers in a productive, collaborative spirit do not always know where to find them.

Math Teachers' Circles (MTCs) are a unique form of professional development that tap into this instinct that mathematicians have to share their love of mathematics with others. In contrast to other important forms of professional development for mathematics teachers that require specialized educational knowledge, MTCs leverage mathematicians' disciplinary knowledge and skill. That is, as mathematicians facilitate an MTC session, they ask probing questions, model mathematical thinking, and encourage mathematical conjecturing, exploration, communication, and discovery. This is similar in many ways to other activities that mathematicians regularly engage in, such as training Ph.D. students, guiding research experiences for undergraduates, or leading a capstone experience for mathematics majors. This makes MTCs an accessible entry point for mathematicians interested in working with teachers.

In this article we describe the history of MTCs, the MTC model, a sample session, and what we know about the impact of MTCs on teachers, mathematicians, and the K–20 mathematical community.

The Rise of MTCs

The idea for Math Teachers' Circles originated from a middle school mathematics teacher, Mary Fay-Zenk, at the time a teacher at Cupertino Middle School in Cupertino, CA. Intrigued by the mathematics at the San Jose (CA) Math Circle for students, Fay-Zenk thought that teachers should have their own outlet for mathematical exploration. She and a small group of other teachers and mathematicians, including Tom Davis (co-founder

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Brianna Donaldson is director of special projects at the American Institute of Mathematics. Her email address is `brianna@aimath.org`.

Michael Nakamaye is professor of mathematics and statistics at the University of New Mexico. His email address is `nakamaye@math.unm.edu`.

Kristin Umland is associate professor of mathematics and statistics at the University of New Mexico. Her email address is `umland@math.unm.edu`.

Diana White is associate professor of mathematics at the University of Colorado Denver. Her email address is `diana.white@ucdenver.edu`.

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of Silicon Graphics), Wade Ellis (emeritus from West Valley College), Tatiana Shubin (San Jose State University), Sam Vandervelde (now at St. Lawrence University), and Joshua Zucker (now at the American Institute of Mathematics), developed this idea over the course of about a year. They secured sponsorship for a summer workshop at the American Institute of Mathematics (AIM) in 2006, thereby launching the first MTC. It continues to thrive, with some participants having attended throughout the eight years since its inception.

In 2007 AIM began to organize annual summer training workshops, entitled "How to Run a Math Teachers' Circle," to support the spread of the model. The fifteen workshops held to date have resulted in the formation of almost seventy new MTCs in thirty-seven states. As each MTC typically involves between 15 and 20 teachers during a normal year, most at the middle school level, collectively these MTCs reach 1,000 to 1,500 teachers per year. The MTC Network (`http://www.mathteacherscircle.org`) links together MTCs around the U.S., providing ongoing support in the form of materials from successful math sessions, discussion groups, and a semi-annual newsletter. The program has garnered support from the National Science Foundation (NSF), National Security Agency, American Mathematical Society (AMS), Mathematical Association of America (MAA), Math for America, Educational Advancement Foundation, and private donors. In its recent Mathematical Education of Teachers II (MET2) document [C], the Conference Board of the Mathematical Sciences (CBMS) recommended the MTC model among a handful of professional development models for middle and high school teachers.

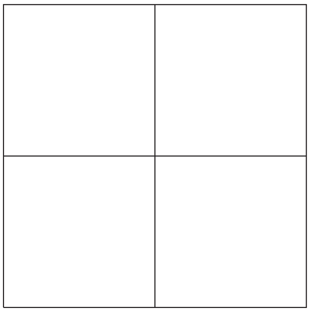
The MTC Model

Math Teachers' Circles focus on highly interactive sessions in which teachers work with mathematicians on rich mathematical problems. Typically, a new MTC group begins with an intensive summer workshop for between fifteen and twenty-five middle school teachers who immerse themselves in mathematics full-time for a week guided by three or four mathematicians. During subsequent academic years, the group continues to meet approximately once a month, with summer workshops held as necessary to involve new teachers. Needs of local teachers drive the logistics of how the model is implemented at different sites, but all MTCs focus on problem solving in the context of significant mathematical content, draw on the content expertise of mathematicians, and have high levels of interactivity among participants and session leaders.

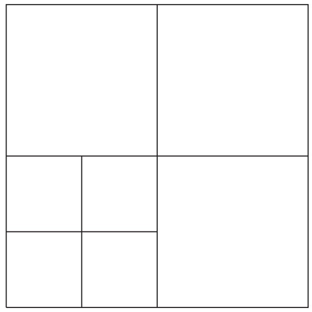
Two primary goals of the MTC model are to encourage teachers to develop as mathematicians by engaging them in the process of doing mathematics and to create an ongoing professional K–20 mathematics community. The core model does not explicitly address enhancing instruction, although many MTC groups choose to include formal discussions of pedagogy as well as informal time to discuss classroom ideas and concerns. Thus, MTCs are not intended as a comprehensive form of teacher professional development. Rather, they fall on the mathematical end of the spectrum, and they complement other forms of professional development that are more focused on direct classroom applications. As a result, it is natural to pair MTCs with other kinds of professional development, such as classroom coaching, lesson study, or Professional Learning Communities.

The problems that form the basis of MTC activities are mathematically rich and have multiple entry points or methods of attack, thereby encouraging collaboration and communication among group members with varying skill levels and backgrounds. Since participants do not know the method of solution in advance, the problems require creativity and perseverance. To draw participants in, session leaders try to choose problems that are immediately appealing in some way. The work that participants do to solve the problems leads them to methods and techniques that are broadly applicable in other mathematical endeavors, such as considering a special case or a more general case or drawing a picture or thinking abstractly or one of the many other techniques familiar to good problem solvers. The pursuit of the solution also reveals specific mathematical content. Those readers familiar with the Common Core State Standards for Mathematical Practice will recognize that this focus of the MTC model on doing interesting mathematics presents an opportunity to help teachers develop a deeper understanding of how the mathematical practices interact with mathematical content.

The initial leadership team for each MTC generally consists of two mathematicians, two middle school math teachers, and an additional organizer with expertise such as well-developed connections to the local teaching community or grant-writing experience. These leaders are genuine partners in planning and developing their MTC to meet the needs of the local mathematical and teaching community, with each person bringing his or her distinct skills to the table. In a typical case, the mathematicians provide leadership in choosing rich material and facilitating the teachers' work during the sessions, the teachers help design the program so that it is an appealing and meaningful experience for their colleagues, and the additional



Four Squares



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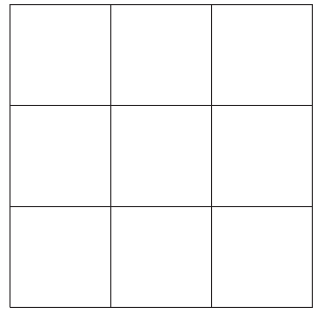

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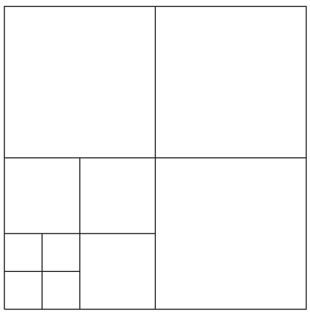

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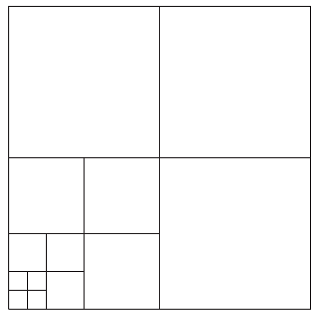


Thirteen Squares

organizer helps connect the team with resources for recruiting participants and raising funding. They all work collaboratively on developing a vision for their MTC, planning workshops and meetings, and fostering a positive atmosphere for all involved.

To provide a better understanding of what an MTC session entails and the kind of mathematical exploration and dialogue MTCs can support, we now turn our attention to an example of the type of problem that teachers and mathematicians collaborate on in a typical meeting.

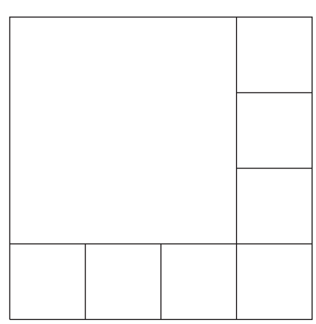
An MTC Session: Dividing Squares

The session begins by showing teachers some pictures of a square divided into smaller squares, not necessarily of the same size. Teachers work for about fifteen minutes creating different divisions of a square and recording their results. After sharing, two natural questions about this division arise:

1. For which whole numbers n is it possible to subdivide a square into n smaller squares?
2. For which whole numbers n is it *not* possible to subdivide a square into n smaller squares?

For the first question, the examples teachers produce seem to indicate that when $n \geq 6$ there is a division into n squares. Teachers then identify, possibly with some guidance from the session leader, some nice relationships between their different pictures. For example, the pictures above with four and seven squares are closely linked: these are the same except that one square has been further subdivided into four smaller squares. This is a process which can be repeated, and each time this is done, three new squares are added—since a single square is replaced with four smaller ones. Above are pictures showing two further divisions starting from the decomposition into seven squares.

Teachers observe that these divisions give $4, 7, 10, 13, 16, \ldots$ squares. This is an example of an arithmetic sequence generated by a geometric process. The same operation, applied to the "tic tac toe" division into nine squares, produces examples with $9, 12, 15, 18, \ldots$ squares. At this point, most of the missing numbers belong to another arithmetic sequence, namely $5, 8, 11, 14, \ldots$. Teachers can now apply this technique to a division into eight squares, as shown below:



Putting all of these examples together gives divisions of the square into $4, 7, 8, 9, 10, 11, \ldots$ smaller squares. The original square itself, without division, is one square. In our experience, teachers come up with divisions into these numbers, up to about thirty, and also identify patterns which can be used to generate sequences. The session leader often needs to provide some guidance putting together the different ideas and constructions if a formal argument, as presented here, is a major goal of the session.

The only remaining possibilities are $2, 3, 5$, and 6 squares. Dealing with these special cases can require some prompting from the session leader, particularly the cases of 2 and 3, which appear very simple. A good argument to eliminate these is to note that, as soon as a division is made, each of the four corners of the initial square must belong to a *different* smaller square. The session leader must decide how much to stress this argument, as it does not always come naturally to teachers and is not the main focus of the problem. Five squares is also not possible, but the reason is more challenging to explain: the four squares containing the corners of the original square cannot be the same size in this case, but it is impossible for the complement of these four squares to be a square, since this complement is not a convex set. This is subtle and necessarily so because it establishes that *regardless of how we cut*, we will never arrive at exactly five smaller squares. Here too the session leader must decide how much rigor to pursue in the argument. Cutting into six squares is possible and is left to the reader. The conclusion of this first investigation answers questions 1 and 2: it is possible to cut the large square into any number of smaller squares with the exception of $2, 3$, and 5. This first part of the session can take anywhere from forty-five minutes to an hour and a half depending on the level of precision and thoroughness of the arguments.

One of the hallmarks of Math Teachers' Circle investigations is beginning with a simple, readily accessible question and then heading as far as the mind and imagination are willing to go. In this example, there are many further questions which can be asked about the process of subdividing a square into smaller squares. In the Albuquerque Math Teachers' Circle session described here, the leader posed the following question:

3. Is it possible to perform the division so that there is a *unique* square of smallest size?

The open-endedness makes this a challenging question, and intuitions for the answer vary. In the session described here, one of the teachers was able to successfully answer this question, but, in general, the session leader needs to decide whether to provide guidance or to leave teachers with an open question. This line of inquiry can be carried even further:

4. Is it possible to perform the division in such a way that no two squares have the same size?

Another direction of inquiry would go in the opposite direction: given a certain number of squares (of various sizes) is it possible to assemble them to form a single square? The question about squares can also naturally be adapted for different shapes. One very famous example of decomposing an equilateral triangle into smaller equilateral triangles is Sierpinski's triangle.

A substantial amount of research[1] has been done on the question of decomposing a rectangle into squares [D], [BSST], [G]. However, the teachers are not familiar with the details of that mathematics, and unless this happens to be directly in the research area of the mathematicians, they will not be familiar with it either. Thus, together they are participating in their own miniresearch experience as they explore this topic.

This problem illustrates several features of typical MTC problems. They are easy to state and require minimal background knowledge, allowing everyone to immediately engage in a search for an answer. The questions considered are often open ended, providing teachers an invaluable opportunity to experience problem solving as mathematicians. They also lead to rich and interesting investigations, leaving the door open for teachers to continue their mathematical explorations long after the MTC session has ended.

MTC Problems Can Lead Anywhere

One session of a Math Teachers' Circle in the San Francisco Bay Area led to sustained collaboration among mathematicians, teachers, and students. In this session, teachers were given a few minutes to make their best guess as to the size of 100! with the answer expressed in scientific notation. After discussion, the session continued with a goal of providing an estimate that is accurate to within a factor of 2. A good starting point for making these estimates is that $2^{10} = 1024$, which is pretty close to 10^3; from here it follows that 2 is about $10^{0.3}$. We can deduce from here that 5 is about $10^{0.7}$. Using these estimates and the fact that 80 and 81 are neighboring integers whose only prime factors are 2, 3, and 5, we can deduce that 3 is about $10^{0.48}$. These estimates for 2, 3, and 5 can be used directly to give estimates, as a power of 10, for many of the numbers between 1 and 100.

[1] *The authors are grateful to Jim Madden for sharing the rich history of this problem with them.*

More work is then needed to fill in numbers with prime factors other than 2, 3, or 5.

Critical in this method was finding the numbers 80 and 81, which are adjacent integers with small prime factors. Intrigued, the session leader had two high school students from Morgan Hill, CA, Mark Holmstrom and Tara McLaughlin, investigate these special numbers further. This led to a method of constructing large neighbors out of sets of small neighbors. In the end, around 345,000 neighbors were identified with prime factors less than 200, the largest having twenty-three digits! Holmstrom and McLaughlin's project won first prize in the regional science fair and a trip to Pittsburgh for the Intel International Science and Engineering Fair, where they received an honorable mention from the AMS. The two students and the session leader wrote up these results, and they will be published in *Experimental Mathematics*.

On at least two other occasions, original mathematical research has stemmed from MTC sessions and been published. One of these cases [A] involved research developed by the mathematical facilitator and a collaborator. The other example, involving the geometry of the card game Set, recently appeared in an article [B] in the *College Mathematics Journal* of the Mathematical Association of America. It is coauthored by ten teachers and three facilitators!

Imagine the impact on these teachers' self-confidence and mathematical self-identities that comes from contributing in a bona fide way to genuine mathematical research!

Focus on Partnerships

Whereas the organization of each local MTC varies considerably, core principles persist across locales. One of the key principles involves cultivating cross-community partnerships. As noted earlier, the leadership team should consist of mathematicians, teachers, and administrators. Additionally, MTC sessions themselves often find a variety of partners coming together to engage in mathematics. Indeed, depending on the local chapter, MTCs can include not only the mathematician facilitators and middle school teachers but also some elementary or high school teachers, other mathematics or science faculty, and undergraduate or graduate students training to be teachers.

The CBMS MET2 document cited this vertical integration as follows: "A substantial benefit of [Math Teachers' Circles] is that they address the isolation of both teachers and practicing mathematicians: they establish communities of mathematical practice in which teachers and mathematicians can learn about each other's profession, culture, and work" [C].

Impact on Teachers

Ongoing research demonstrates the benefits of MTCs for teachers' confidence, knowledge, and teaching of mathematics. A team led by Diana White has recently reported [W] that teachers' mathematical knowledge for teaching (MKT) increases significantly over the course of a four- to five-day intensive summer MTC workshop. Previous research has found a positive correlation between teachers' MKT and student achievement scores [H]. Additionally, researchers from the University of Colorado at Colorado Springs have reported evidence from classroom observations that MTC participants significantly increase the use of inquiry-based teaching practices over a one-year period [M].

The coauthors of this article are members of a research team that is investigating the impact of MTCs with support from the NSF. Our research has three primary components: a further investigation into the effects of MTC participation on teachers' mathematical knowledge for teaching, a series of annual surveys aimed at learning more about the teachers who participate and their motivations for doing so, and the development of in-depth, multiyear case studies that include classroom video observations and interviews. The case study results are too preliminary to report here, but the surveys are providing an interesting picture of teachers' experience with MTCs. For example, on the 2012 survey, 75 percent of the 169 respondents said that MTC participation has affected their teaching. Examples ranged from specific topics that MTC sessions have helped them teach more effectively (e.g., "I've incorporated number theory ideas I learned with respect to comparing fractions") to broader changes in how they ask students to approach mathematics (e.g., "I give students fewer steps and ask them to tackle complex problems with less scaffolding of the problem itself. I also have been helping students reflect on what their problem-solving process is and how to optimize their problem-solving process"). Over half of the respondents told us that MTC participation has changed their view of mathematics. They cite that MTCs enable them to see deeper connections between mathematical topics, expose them to fields of mathematics that they were previously unfamiliar with, and help them envision doing mathematics as a more collaborative and enjoyable enterprise. Approximately half of the respondents also wrote that participating in MTCs has affected their other professional activities, for example, leading them to become more involved in collaborations, leadership, and student extracurricular mathematics activities.

Perhaps the most rewarding result we have seen, and the one that we believe will ultimately have the biggest impact, is that participating teachers report beginning to see themselves as mathematicians. For example, here are a few quotations from middle school teacher participants:

- "You encouraged me as a mathematician. I have never actually seen myself as one before."
- "I've become much more of a recreational problem solver—I always have my problem notebook with me for down times."
- "MTC grows teachers who are also mathematicians."

Impact on Mathematicians

There has been considerably more research into the benefits of MTC participation for teachers, but we now turn our attention to how mathematicians benefit from MTCs. Beyond the opportunity to get involved in K–12 education while sharing one's love of mathematics, MTCs can be synergistic with other efforts, such as mentoring student research, as illustrated in the example above. MTCs can also provide an excellent avenue for contributing to one's departmental or institutional mission, especially at a state-supported institution or an institution with a large teacher-training program. For those involved in teacher training or professional development, MTCs can provide a way to maintain a professional connection with the local teaching community and to offer ongoing support to teachers who have participated in other university programs. One mathematician chose to become involved because "we have relationships with many teachers in our surrounding districts. Our teachers are eager for any help we can provide. The Math Teachers' Circle will provide a venue to address their needs." Another writes that "offering Math Teachers' Circles would be a perfect opportunity for our master's program to continue to support the teachers graduating from it, and providing leadership opportunities for them." Having an impact on the preparation of local K–12 students who will eventually attend the university is another potential long-term benefit. For example, another mathematician writes that the primary reason for becoming involved was "to form relationships with local schools while enhancing the instruction provided to future undergraduate students in our target recruitment populations."

Increasingly, MTCs are becoming a professionally recognized forum for mathematicians to contribute to education. The AMS and MAA have supported the effort from early on, and the MAA continues to partner with AIM to host one of the "How to Run a Math Teachers' Circle" training workshops each summer. Anecdotally, more than a handful of MTC leaders have noted that their MTC involvement was a contributing factor in a positive job search experience or tenure or promotion decision. Several MTC leaders have also received partial support for their MTCs as part of the broader impacts of their NSF research grants, indicating a wider recognition that MTCs constitute a valuable contribution that mathematicians can make to society.

Final Thoughts

By their very nature, Math Teachers' Circles are a natural forum for mathematicians and mathematics departments to reach out to practicing teachers and form lasting relationships that recognize the common purpose which unites K–20 educators. Some faculty who participate in MTC meetings are inspired to seek other ways to contribute to the larger teaching community. This may include working with preservice teachers at their institution or becoming more involved with state or national educational efforts. Whether it serves as a launching point for deeper involvement with teacher education or as an end in itself, participation in an MTC can impact both teachers and students and can play an important and fulfilling role in the professional lives of many mathematicians.

Acknowledgment

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References

[A] G. Abrams and J. Skylar, The graph menagerie: Abstract algebra and the mad veterinarian, *Mathematics Magazine* **83** (3) (2010), pp. 168–179.

[B] M. Baker, J. Beltran, J. Buell, B. Conrey, T. Davis, B. Donaldson, J. Detorre-Ozeki, L. Dibble, T. Freeman, R. Hammie, J. Montgomery, A. Pickford, and J. Wong, Sets, planets, and comets, *The College Mathematics Journal* **44** (4) (2013), pp. 258–264.

[BSST] R. L. Brooks, C. A. B. Smith, A. H. Stone, and W. T. Tutte, The dissection of rectangles into squares, *Duke Mathematical Journal* **7**, (1) (1940), pp. 312–340.

[C] Conference Board of the Mathematical Sciences, *The Mathematical Education of Teachers II*, American Mathematical Society, Providence, RI; and Mathematical Association of America, Washington, DC, 2012.

[D] M. Dehn, Über Zerlegung von Rechtecken in Rechtecke, *Mathematische Annalen* **57** (1903), pp. 314–332.

[G] D. Gale, More on squaring squares and rectangles, *Mathematical Intelligencer* **15** (4) (1993), pp. 60–61.

[H] H. C. Hill, D. L. Ball, M. Blunk, I. M. Goffney, and B. Rowan, Validating the ecological assumption: The relationship of measure scores to classroom teaching and student learning, *Measurement: Interdisciplinary Research and Perspectives* **5** (2-3) (2007), pp. 107–117.

[M] P. D. Marle, L. L. Decker, and D. H. Khaliqi, *An inquiry into Math Teachers' Circle: Findings from two year-long cohorts*, paper presented at the School Science and Mathematics Association Annual Convention, Birmingham, AL, 2012.

[S] T. Shubin, Math circles for students and teachers, Mathematics Competitions, *Journal of the World Federation of National Mathematics Competitions* **19** (2) (2006), pp. 60–67.

[W] D. White, B. Donaldson, A. Hodge, and A. Ruff, Examining the effects of Math Teachers' Circles on aspects of teachers' mathematical knowledge for teaching, *International Journal of Mathematics Teaching and Learning*, published online September 26, 2013, at `http://www.cimt.plymouth.ac.uk/journal/`.