Chapter 5
Teacher Knowledge and Visual Access to Mathematics

Jill Neumayer DePiper and Mark Driscoll

Abstract We propose that there exists mathematical knowledge for teaching (MKT) specific to visual representations (VRs), abbreviated MKT-VR. We define a VR as a graphic creation, such as a diagram or drawing, which illustrates quantities and shows quantitative relationships or which illustrates geometric properties of figures and shows geometric relationships. A teacher with strong MKT-VR will, for example, be able to use and understand VRs in his/her own problem solving and will have mathematical knowledge specific to teaching students to use, analyze, and solve problems with VRs. The Visual Access to Mathematics (VAM) project seeks to help teachers understand the value of VRs, specifically when teaching and learning ratio and proportional reasoning content. This chapter lays out a theoretical framework that we anticipate using to guide and benchmark future research.

Keywords PCK · Mathematical knowledge for teaching · Visual representations · Student thinking · Teacher knowledge · Proportional reasoning

5.1 Introduction

The chapter describes a construct of mathematical knowledge for teaching (MKT) specific to visual representations (VRs), abbreviated MKT-VR, focused on the context of ratio and proportion in particular. We detail why MKT-VR related to ratio and proportional reasoning is important for middle grades mathematics teachers, and how it relates to what is important for students. Following our discussion of the importance of MKT-VR, we describe our study on supporting teachers' MKT-VR in teacher professional development, and how we are measuring teacher MKT-VR related to ratio and proportional reasoning. Beyond the topic of ratio and proportional reasoning, we argue for an interest in studying MKT-VR across school mathematics. This would be consistent with growing evidence of the instructional

J. N. DePiper (✉) · M. Driscoll
Education Development Center, Inc., Waltham, MA, USA

S. M. Uzzo et al. (eds.), *Pedagogical Content Knowledge in STEM*,
Advances in STEM Education, https://doi.org/10.1007/978-3-319-97475-0_5

efficacy of visual representations for all mathematical content from kindergarten into high school.

5.2 Why Proportional Reasoning?

Middle grade mathematics sits at the crossroads in a student's school mathematics journey: it marks the end of focused attention on number and operations, and it comes before other themes, such as geometry and measurement, which are explored in depth in high school. While middle grades mathematics classes are not always organized around a theme, proportionality is an overarching concept in these grades, and "one that unites, relates, and clarifies many important middle grades topics" (Lanius and Williams 2003, p. 392). Proportionality has been called the "cornerstone of higher mathematics and the capstone of elementary concepts" (Lesh et al. 1988, p. 98). As the capstone to elementary topics, proportionality as a concept builds from understandings about number and operations and invites connections to real-world situations and to variation. In mathematics in high school and beyond, during more in-depth studies of algebra, probability, geometry, and measurement, understanding of proportionality is a prerequisite, as relationships between quantities are key to functions and variation. Proportionality also has many important connections outside of mathematics. Proportional literacy serves all citizens well, whether it is by using unit rates to compare grocery prices, understanding the relevance of growth rates in measuring economic health, or being aware of how population proportions influence political decisions.

Proportional reasoning, including the study of proportionality, refers to a mathematical way of thinking, specifically: "in which students are solving problems about proportional situations: Proportional *reasoning* refers to detecting, expressing, analyzing, explaining, and providing evidence in support of assertions about proportional relationships" (Lamon 2007, p. 647). Reasoning proportionally does not mean just using a cross-product approach for solving problems but includes understanding *when* problems are proportional or not and *wha*t the situations mean. Proportionality may be illustrated: in descriptions of how quantities vary, such as "Mark earns $85 every two weeks…"; algebraically, as in linear functions, $y = mx$; or geometrically, as a line that passes through the origin (Lanius and Williams 2003). Also foundational to proportional reasoning, students need to understand ratios both as *composed units*—e.g., that in the ratio 1:4, for every one unit of one quantity, there are four units of another quantity—or as *multiplicative comparisons*, where there are four times as many of the second quantity as the first (Lobato and Ellis 2010). Understanding proportionality, its applications, and how to reason proportionally are foundational to middle grades' students' further studies, and attention to these areas can engage students in key mathematical content and practices.

5.3 Tackling the Complexity of Proportional Reasoning with Visual Representations

Using visual representations (VRs) can support learners' understanding of these key mathematical concepts and practices. VRs are a graphic creation, such as a diagram or geometric drawing, which illustrate quantities and/or geometric properties and show relationships among quantities and/or geometric figures[1]. Examples of VRs in rational number and proportional reasoning contexts are number lines and rectangular tape diagrams (sometimes called "strip diagrams"). Research specifically recommends VRs to reinforce students' conceptual understanding of rational numbers (Gersten et al. 2009; Siegler et al. 2010). A VR can call students' attention to the quantities presented in a problem and the relationships *between* quantities. VRs can also scaffold students' understanding of the symbolic approach to the problem (e.g., Siegler et al. 2011), through, for example, presenting equipartitioning and highlighting relations among fractions, decimals, and percentages (Gersten et al. 2009).

As a tool, VRs can support problem solving, communication, and engagement. Using a VR can support students in making sense of the problem, and then subsequently make modifications in light of sense making, and select a solution strategy (Ng and Lee 2009). Visual representations can provide a bridge from text to arithmetical or algebraic representations, which is a valuable support for all students and invaluable for students who are English Learners. VRs help students by linking the relationships between quantities in the problem with the mathematical operations needed to solve the problem. Opportunities to use mathematical visual representations provide students access to mathematics, support their engagement in problem solving, facilitate communication of their mathematical thinking, and develop the mathematical practices outlined in the CCSSM.

More generally, VRs are an element of multimodal communication. Multimodal mathematical communication refers to the various ways in which students convey their mathematical thinking, including language, gestures, drawings, or the use of tools (e.g., physical models, manipulatives, and technology). Students may use a combination of modes at once or different ones in isolation. To enhance mathematical learning opportunities for all students, particularly ELs and those struggling with language, research stresses the importance of creating classroom environments that encourage multimodal communication (Chval and Khisty 2001; Khisty and Chval 2002; Moschkovich 2002). Such environments can foster the development of the Standards for Mathematical Practice (SMP) that are articulated in the Common Core State Standards for Mathematics (CCSSM, CCSSM SMP; NGA & CCSSO 2010) by providing students access to the mathematics, helping them construct viable arguments, and providing opportunities to attend to precision by developing accurate mathematical language.

[1] This definition of a mathematical visual representation is particularly germane to number and algebra contexts.

A VR can also be a powerful tool toward eliciting the use of thinking aligned with the Standards for Mathematical Practice (SMPs). As much of middle-grade mathematics relates to quantities, spatial properties, and related problems solving, VRs can support learners in identifying and making sense of these quantities, properties, and relationships in ways that align with SMP 2 (*reason abstractly and quantitatively*) and SMP 7 (*look for and make use of structure*). Quantitative reasoning involves reasoning about the relationships among quantities and does not necessarily need to involve algebraic expressions or assigning variables (Smith and Thompson 2007). SMP 2 emphasizes rich conceptual understandings and not reliance of algorithmic thinking: "Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them" (CCSSI 2010, p. 6). SMP 2 specifically emphasizes the ability to decontextualize and contextualize when using mathematics to solve problems (NGA and CCSSO 2010), and a VR can present mathematics in ways that a symbolic representation does not. For example, suppose a word problem says that *Maria has $10 more than Albert, and together they have $40*. We can write a symbolic representation of this situation, say, $M + A = (A + 10) + A = 40$, with M and A representing, respectively, Maria's amount and Albert's amount. In doing so, we have *decontextualized* the situation, that is, abstracted the quantitative information. VRs can represent this same information, support the decontextualizing and recontexualizing, and provide an artifact for that conversation. Using structure (SMP 7) is emphasized in proportional reasoning contexts, such as when learners use a double number line to note how, as two proportional quantities covary, the ratio between them remains invariant.

5.4 Student and Teacher Understanding of the Importance of Visual Representations

While VRs and opportunities to use them in mathematical thinking and learning can support mathematical reasoning, learning how to use a visual representation as a tool is a skill, in and of itself. Unlike geometry tasks, tasks where quantitative reasoning is prominent, such as algebraic word problems, usually do *not* provide visual representations. In such cases, knowing how to draw one's own visual representations is a very valuable skill. The 2012 IES Practice Guide, *Improving Mathematical Problem Solving in Grades 4–8*, based on an examination of hundreds of relevant, rigorous studies, recommends teaching students how to use VRs to enhance their mathematical problem solving: "Students who learn to visually represent the mathematical information in problems prior to writing an equation are more effective at problem solving" (Woodward et al. 2012, 23). It is important to study and become proficient with a variety of visual representations and to understand how to select the representations most appropriate for solving a task (Woodward et al. 2012). This "includes knowing what particular representations are able to illustrate or explain, and to be able to use representations as justifications for other claims" (Zbiek et al. 2007, p. 1192). Furthermore, the importance of rich understanding of VRs is not limited to middle school students, as research has found that competent mathematical thinkers, in university-level mathematics, use VRs flexibly in problem solving (Stylianou 2002; Stylianou and Silver 2004).

The ability to interpret and construct various mathematical representations, and to change representations appropriately, is considered *representational fluency* or *diagram sense*. A learner needs to develop diagram sense—knowing when various VRs are most useful—for example, knowing that tree diagrams can help organize probabilistic thinking, that number lines are often handy for rational number tasks, and that tape diagrams can propel thinking about algebraic word problems. We know that learners need to develop number sense in order to judge the reasonableness in their own and others' calculations. We also know that number sense can be learned through ample opportunities to reason about numbers and operations. Similarly, we believe that *visual representation sense* can be learned through ample opportunities to represent problems and to reason with the diagrams.

A critical piece in supporting students' representation fluency and fostering a culture of multimodal communication and engagement in the classroom in these ways is for teachers to value and encourage the use of mathematical VRs, especially as tools for reasoning and communicating mathematical thinking. For example, students need a diet rich with representations and an opportunity to study a variety of VRs to be able to understand how to select the representations most appropriate for solving a task (Woodward et al. 2012). In our experiences across multiple studies on teacher instruction, however, we have observed that students' opportunities for reasoning with and about quantities and relationships too often take a back seat to opportunities for students to practice computational procedures. The complex nature of proportional relationships is not consistently tackled in instructional situations. Without attention to conceptual understanding of relationships, learners may struggle to identify rates and ratios, reason about variation, or follow changes in units or analysis. Students may rely on algorithmic thinking, such as cross products, and not engage in reasoning proportionally, which does not strengthen their understandings of these key ideas.

Research has found that teachers may be less prepared in some mathematics content, as compared to other content areas, and specifically less prepared in the key areas of fraction and proportional reasoning and representing relationships (Siegler 2011). In reasoning proportionally, teachers tend to rely on algorithmic thinking, such as the cross-multiplication algorithm in proportional situations (Orrill and Brown 2012; Riley 2010; Singh 2000), and may be using cross multiplication as a procedure without conceptual understanding. Teachers often focus student attention on operational understandings and algorithmic thinking, instead of developing their conceptual understandings of proportional reasoning (Lamon 2007). Lack of understanding of content may limit teachers' ability to teach critical mathematics content: "Researchers typically do not associate reasoning with rule-driven or mechanized procedures, but rather with mental, free-flowing processes that require conscious analysis of the relationships among quantities" (Lamon 2007, p. 647). Teachers need a rich understanding of mathematics content, many times noted as *specialized* mathematical knowledge for teaching (Ball et al. 2008; Ma 1999), and also understandings of visual representations in proportional reasoning, similar to how visual representations support student learning in these areas.

Evidence from our previous research from the IES-funded project, Mathematics Coaching for Supporting English Learners (MCSEL)[2] and from others (Stylianou 2011) suggests that US teachers in elementary and middle grades generally are neither experienced nor skilled in understanding and using VRs in mathematics. From analysis of teacher instruction and self-reflection, we identified a critical need in continuing to build teachers' knowledge of how to use diagrams in their own mathematical problem solving. MCSEL developed and studied professional development (PD) for middle grades mathematics teachers of ELs that emphasized the use of visual representations integrated with language support strategies. As we introduced instructional activities to support students in using a visual representation when they approach tasks, we found that teachers also needed support in learning how to use visual representations in their own mathematical problem solving. Related research has also found that while teachers had important knowledge about proportions, their understanding of representations was not coordinated with their understanding of proportions (Orrill and Brown 2012). Participants in teacher professional development were found to rely on addition and subtraction strategies, rather than multiplicative reasoning, and initially struggled with a double number line representation (Orrill and Brown 2012). This research supports these findings, and along with other literature, this suggests that teacher professional development needs to focus on building connections between representations and ratio/proportional reasoning concepts and content.

Finally, we have also been struck by a common perception about VRs exhibited both by students and teachers—namely, that the primary purpose of a VR is to present the *product* of one's thinking about a mathematical task. No doubt this is a helpful role for VR, but it is far from the only, or even most valuable, purpose of creating a VR (e.g., Stylianou 2011). Rather, VRs can be really effective *reasoning tools* for learners trying to solve challenging mathematical tasks. This is a quality that makes them especially valuable for ELs. VRs are also *communication tools*, as students can use them to share their thinking and reasoning, and teachers can use students' VRs to prompt them to discuss relationships between quantities. VRs, far from being just a product of student thinking, can be part of the process of mathematical thinking, reasoning, and communicating.

[2] The Mathematics Coaching for Supporting English Learners research was supported by the Institute of Education Sciences, US Department of Education, through Grant R305A110076 to the Education Development Center, Inc. The opinions expressed are those of the authors and do not represent views of the Institute or the US Department of Education.

5.5 Teacher Professional Development Focused on VRs and Proportional Reasoning Content

The Visual Access to Mathematics (VAM) project seeks to help teachers understand the value of VRs, specifically when teaching and learning ratio and proportional reasoning content. The NSF-funded VAM project[3] is a multi-year design and development project that includes the development, facilitation, and related research on teacher professional development. It seeks to advance knowledge in the field about supporting mathematics teachers of students who are English Learners (ELs) by developing and studying a 60-hour blended-learning professional development (PD) program for middle-grade mathematics teachers who teach ELs. Concentrating on ratio and proportion and related rational number concepts, and relying heavily on technology-supported artifacts of student thinking, the VAM professional development (VAM PD) helps mathematics teachers of ELs become better at making, using, and analyzing VRs for mathematical problem solving, with the goal of improving teacher knowledge and practice. Embedding opportunities for students and teachers to use VRs in classroom environments in combination with teacher attention to and use of students' thinking to create equity in mathematics instruction maximize the value of VRs. The VAM PD blends online and face-to-face components; it includes a summer institute, eight online sessions, and two face-to-face workshops. Both online and face-to-face sessions include activities that focus on developing teacher knowledge in four areas: teacher knowledge of visual representations for problem solving, mathematical knowledge for teaching ratio and proportional reasoning, analysis of student mathematical thinking, and instructional planning with visual representations and language access and production strategies.

As we develop the VAM PD, we are also studying if and how it supports teacher knowledge, analysis of student work, and instructional planning. We hypothesize that developing teachers' abilities to use VRs for problem solving can improve teachers' mathematics instruction and provide ELs greater access to learn productive mathematical reasoning. Two overarching questions guiding this work are: (1) What supports will allow mathematics teachers to develop the pedagogical content knowledge they need to support ELs in mathematical problem solving? and (2) What is the effect of VAM PD on teachers' pedagogical content knowledge about using VRs to support mathematical problem solving? Key elements of the teacher intervention, teacher outcomes, and classroom/student outcomes are presented in the Theory of Change (Fig. 5.1). The focus in this PD and the related research is on the relationship between the VAM intervention and the teacher outcomes, as noted by the bold arrow in Fig. 5.1.

[3] The Visual Access to Mathematics project is supported by the National Science Foundation under Grant No. DRL 1503057. Any opinions, findings, and conclusions or recommendations expressed are those of the author and do not necessarily reflect the views of the National Science Foundation.

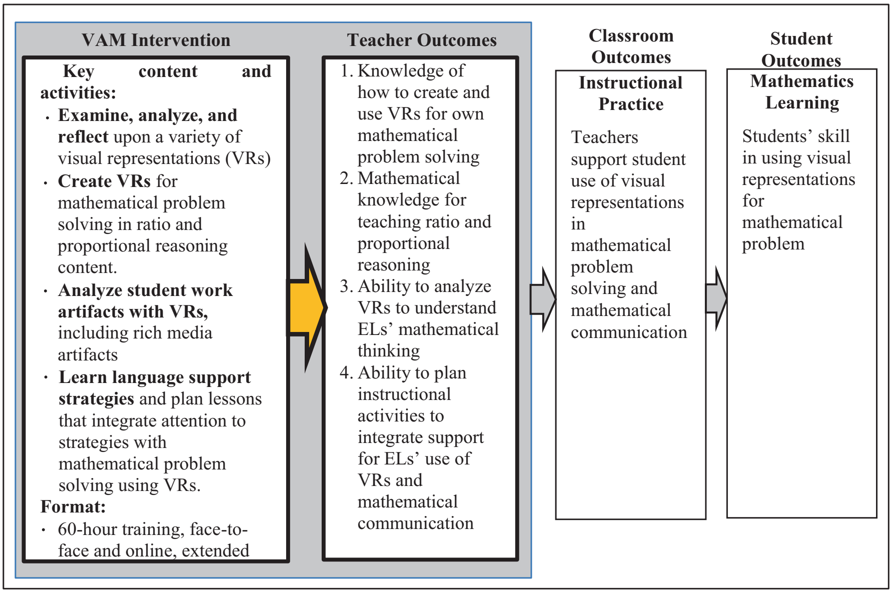


Fig. 5.1 VAM theory of change

5.6 Defining Teacher Knowledge of VRs

In the VAM project, there are multiple ways in which we are seeking to understand the relations between participant experiences and their shifts in knowledge and practice. Specific to our goals of measuring teacher knowledge, we argue that there is a body of knowledge and set of skills associated with fluent use of VRs in mathematics learning and teaching, and we posit that VAM PD will promote such knowledge and skills (Teacher Outcome 1). Our two different target sub-outcomes related to Teacher Outcome 1 (*Teacher Knowledge of Visual Representations*) further specify our interests in teacher learning: (a) improved the ability to use VRs to represent and solve ratio and proportional reasoning problems and (b) improved the ability to evaluate the strengths and limitations of different solutions involving VRs. It is important to revisit the rich educational research in the area of teacher knowledge as we define what teacher knowledge is as related to visual representations.

Thirty years ago, it was groundbreaking when Shulman (1986) placed a focus on the critical role of content knowledge in pedagogy, defining pedagogical content knowledge and identifying major categories of teacher knowledge that were content specific. As many others have expanded on this work, in mathematics education, Ball and colleagues (2005, 2008) have sought to define the knowledge needed by teachers, in order to better understand, measure, and support teacher knowledge. It is their definition that aligns with our current attention to teacher knowledge of VRs. Their working definition of mathematical knowledge for teaching is "the mathematical knowledge that teachers need to carry out their work as teachers of mathematics" (Ball et al. 2008, p. 4).

This attention to the *work* that teachers do as teachers is key in our understanding of teacher knowledge of VRs. As we emphasized in our earlier description of the importance of VRs, VRs support learners' conceptual understandings of content, and understanding in mathematics contexts is content knowledge. As teachers need to know and understand VRs enough to be able to teach them to others and VRs support and are a tool for mathematical thinking and communication, then teachers' VR knowledge is as much a tool for their own problem solving as it is a tool that teachers should teach with and teach their students to use.

5.6.1 Components of MKT-VR

Our VAM team theorizes that teachers need mathematical knowledge for teaching related to visual representations (MKT-VR), which includes both content knowledge and pedagogical content knowledge, situated in the *work* of teaching of middle grades mathematics. In extending the Ball et al. (2008) framing of mathematical knowledge for teaching (MKT) to teaching and learning with VRs, we suggest that a teacher with MKT-VR will be able to use and understand VRs in his/her own problem solving and will have mathematical knowledge specific to teaching students to use, analyze, and solve problems with VRs. A teacher with strong MKT-VR will integrate VRs into their teaching, will be able to generate inferences about students' thinking from students' VRs, and will consistently consider how to promote student use of VRs. Teachers need MKT-VR to conduct tasks such as identifying correct solutions and solving mathematical problems for themselves. Teachers also need MKT-VR, which includes being able to provide students with explanations for why particular solution strategies work, to diagnose student errors with different strategies and to understand nonstandard yet effective problem solving.

We believe that MKT-VR includes both content knowledge and pedagogical content knowledge related to VRs, and it can be defined broadly as a *body of teacher knowledge and set of skills associated with fluent use of VRs in mathematics learning and teaching*. While other models of MKT (Ball et al. 2008) further subdivide content knowledge and pedagogical knowledge into several subdivisions, at this time, we do not try to mirror these categories in our definition of MKT-VR. Instead, we seek to define the construct, and determine which understandings are key as related to VRs. CK-VR and PCK-VR are distinct but related elements of MKT-VR and related elements of middle-grade mathematics teacher knowledge. We theorize three important categories of MKT-VR, two within the area of content knowledge (CK) and one focused on pedagogical content knowledge (PCK), and we will refine these as we continue to learn from work with participants in the VAM PD.

5.6.2 Content Knowledge Related to Visual Representations (CK-VR)

CK-VR is mathematics knowledge related to classroom instruction, students or teaching, specific to visual representations. While content knowledge for teaching is broader than CK-VR, we identify specific types of content knowledge specific to visual representations. As teachers in our work have engaged with VRs for their own mathematical learning and in classroom instruction, the importance of *alignment* of VRs to their purposes and the importance of *strategic use* of VRs in their work have emerged as two key areas of content knowledge. *Alignment* includes making connections between the mathematics task being considered and the solver's purpose (process or product), where there is attention to alignment with the nature of the task and alignment with purpose for using the VR. *Strategic use* is knowledge about *how* to use a VR in situation.

Alignment VRs can play multiple roles as a tool in problem solving and for communicating mathematical concepts and solutions with others (Stylianou 2011). VRs may support individual cognition by organizing information, recording information and reducing cognitive load, allowing manipulation of information and therefore facilitating exploration, and supporting the problem solver in monitoring progress and approaches (Stylianou 2011). Visual representations can also support communicating mathematically and be a tool in social practice for presenting obvious and not-so-obvious perspectives and information and for allowing the sharing of strategies and negotiation of new ideas (Stylianou 2011). Because of these multiple roles that VRs can play in mathematical problem solving, key knowledge for teachers is an understanding of which VR to use when and how to align a VR to a goal. The goal may be focused on specific mathematics content, such as related to supporting or conveying understanding of unit rates, or it may be about communicating mathematically.

Strategic Use of VRs While a VR can serve multiple roles in mathematical problem solving, all VRs may not be equally effective. In our previous IES-funded MCSEL research, we identified four features of strategic use of VRs for mathematical problem solving[4]:

- Clear representation of the given *quantities* in a problem.
- Clear representation of the *relationships between the given quantities*, including surfacing implicit relationships or attending to proportionality. In ratio and proportional reasoning contexts, given quantities and relationships can be represented on a tape diagram and a double number line, and new quantities and relationships can be added to, or emerge, on a tape diagram and on a double number line to help solve the task.

[4] Here we describe VRs in quantitative settings. For geometric settings, comparable wordings refer to geometric properties and relationships, instead of quantities and quantitative relationships.

- *Strong potential to reveal relationships visually between the given quantities and the goal quantity (connecting a VR to an algebraic or symbolic relationship).* Effective VRs highlight connections between VRs and related symbols/calculations/algorithms.
- *Identification within the VR of features related to important mathematical concepts leading to the* goal. The important concepts could include unit rate or multiplicative comparisons.
- *Clear labels and markings* (such as shading, labels, dotted lines, etc.) that support managing data, recording the flow of reasoning, and representing or communicating the quantities and relationships in the problem.

We argue that the alignment of a VR to the purposes listed above (i.e., understanding information, recording information, supporting exploration and problem solving, presenting perspectives, and sharing strategies) is enhanced through using a VR strategically with attention to the features listed above. For example, it is more strategic to represent quantities clearly and highlight implicit relationships when VRs are used to organize and record information. When VRs are used to share strategies and ideas with others, labels and shading as well as attention to scale in the drawing may be important features of a VR to support its use as both a communication tool and a problem solving tool. A VR that identifies a unit rate or the relationships between quantities may be a strategic use of a VR to support connections to algorithms or calculations and may align with mathematical goals.

These two aspects of MKT-VR are primarily CK-VR, though their implications are not strictly restricted to content. For example, a teacher response (VAM teacher participant, Session 4, November, 2016) presents how content knowledge and pedagogical knowledge intertwine in the areas of *alignment* and *strategic use*:

> I love the idea that one can use two tape diagrams stacked on each other to make concrete visual comparisons of quantities, such as what we observe for the mixing paint tasks. …For the mixture problems, the other key idea to take away is that both tape diagrams must be the same size! (Like a common denominator…

In the response, the teacher appreciates that one can use the area in tape diagrams to compare paint mixture proportions, and notes alignment among task, content, and VR. Specifically, if the task is to decide if 4 parts blue paint mixed with 3 parts yellow paint is darker than a mixture of 5 parts blue to 4 parts yellow, one can divide one tape in a 4:3 ratio and another tape in a 5:4 ratio and see which tape has proportionally more blue. The teacher adds that, to make this area comparison work, the two tapes need to be congruent, so there is a common unit guiding the measures. The response attends to why and how to use a VR and one with a common unit, therefore highlighting strategic use. We hypothesize that VAM PD will promote both types of CK among all teachers: knowledge of the alignment of VRs and how to use a VR strategically in mathematical problem solving.

Pedagogical Content Knowledge Related to Visual Representations (PCK–VR) We argue that a teacher with strong MKT-VR will have not only strong CK-VR but also strong pedagogical content knowledge related to VRs (PCK-VR). PCK includes knowledge of common student errors and misconceptions (or knowledge of students and content); mathematical models, representations, and contexts commonly used by or for students; and the ability to address and understand students' interpretation of mathematics (Ball et al. 2008; Campbell et al. 2014; Hill et al. 2008). We propose that PCK-VR includes these same elements, with an emphasis on VRs. For example, a teacher with high PCK-VR will have strong knowledge of how students think about or learn to use VRs and knowledge of how to teach problem solving with VRs.[5]

5.6.3 Measuring and Assessing MKT-VR

We define MKT-VR to include the sub-constructs CK-VR and PCK-VR, and we have two assessments to measure it. While the VAM project, in which the work to define this construct is currently embedded, does not include assessment development as part of its scope, we are nevertheless developing two exploratory assessments, described below. Assembling and developing our own assessments was necessary as we did not find any available assessments that focused on middle-grades teacher knowledge of visual representations that are appropriate given the scale of this study, where more than 100 teachers will participate in field testing of the PD. For example, previous research on mathematicians' and college students' understandings of VRs (Stylianou and Silver 2004) used interviews and comparisons of VR use between experts and did not focus on the content knowledge needed for the *work* of teaching. While research that involved interviews with teachers solving mathematics tasks with visual representations has provided information on how teachers use and understand VRs (Stylianou 2002), interviews are not appropriate for the scale of our study, and the interviews were not as narrowly focused on teacher knowledge and skills as we are in this work. It is also important to note that research on the importance of VRs for students' problem solving focuses more on the student's correct responses to tasks than it does on mathematical reasoning or communication (Boonen et al. 2014). Boonen et al., for example, make the distinction between a pictorial representation, an inaccurate visual-schematic representation, and an accurate visual-schematic representation. A distinction between a pictorial diagram and the visual-schematic may be too simplified for our purposes, as our participants are drawing diagrams that represent the mathematics (not pictures of the problem context), and we seek to understand the nuances in their ability to do so. In addition, as we are interested in supporting teachers and students in

[5] Ball et al. (2008) write that pedagogical content knowledge may be subdivided into several separate domains, including knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum. In our definition of PCK-VR, we have not tried to define similar sub-constructs. Instead, we have tried to identify and include within a general PCK-VR construct the most important types of pedagogical content knowledge that teachers may need to teach mathematics with VRs effectively.

engaging in the mathematics and gaining access to the task, we recognize that pictorial representations have value for supporting access and language production, and could serve as a tool for student or teacher learning, and thus would treat teachers' knowledge about pictorial representations differently. Given this mismatch in purpose and fit between existing measures related to VRs and our endeavor to understand MKT-VR, we have decided to create two exploratory assessments.

The MKT-VR multiple-choice assessment is based on existing instruments and their items and seeks to measure target outcomes of VAM PD. To assemble the assessment of teachers' MKT-VR, the team has followed general scale development guidelines suggested by DeVellis (2012). Due to project constraints, the team restricted the scale's format to multiple-choice items and used items from established assessments of MKT rather than write original items. After the team developed a list of key skills that the assessment would target, the team reviewed existing test banks and assessments and selected an item pool of 30 multiple-choice items that could potentially measure these skills. Items came from the following sources: the *Learning Mathematics for Teaching* assessment (2009); research by DePiper et al. (2014); the National Assessment of Educational Progress (NAEP, grades 4 and 8); released items from the Educational Testing Service's Praxis tests (focusing on elementary and middle grades mathematics); and released items from the Massachusetts Comprehensive Assessment System (MCAS). Because items were drawn from different sources, the number of distracters associated with each item ranges from 3 to 5. The team tried to generate redundancy of items to measure individual skills (DeVellis 2012).

As there are limits in understanding teacher thinking when using a multiple-choice format instead of an open-response assessment, we have also developed an assessment that is an open-response analysis of VRs that are provided to the respondent. We have other measurement tools within the VAM research plan that will measure other types of teacher knowledge of student thinking and teacher practice in classrooms, related to Teacher Outcomes 2–4 in the Theory of Change (see Fig. 5.1), which will provide different information about teacher PCK-VR and will complement the findings from these multiple-choice and open-response assessments of MKT-VR.

MKT-VR Multiple-Choice Assessment We assembled an assessment using multiple-choice items from other established instruments to measure both of these sub-constructs, CK-VR and PCK-VR. The goal of analysis of the assessment responses will be to draw inferences about teachers' abilities to make and use effective VRs in their own rational number problem solving (CK-VR) and to use VRs for teaching mathematical problem solving as well as for understanding students' mathematical thinking (PCK-VR). We seek to draw inferences about CK-VR and PCK-VR as separate sub-constructs and about MKT-VR as a whole.[6]

[6] Our overall project hypotheses are the following: Compared to control group teachers, teachers who participate fully in VAM PD will demonstrate higher (1) MKT-VR, (2) MKT related to ratio and proportion, (3) ability to analyze student thinking, and (4) ability to plan lessons and activities that integrate VRs and language strategies in mathematical problem solving. The MKT-VR assess-

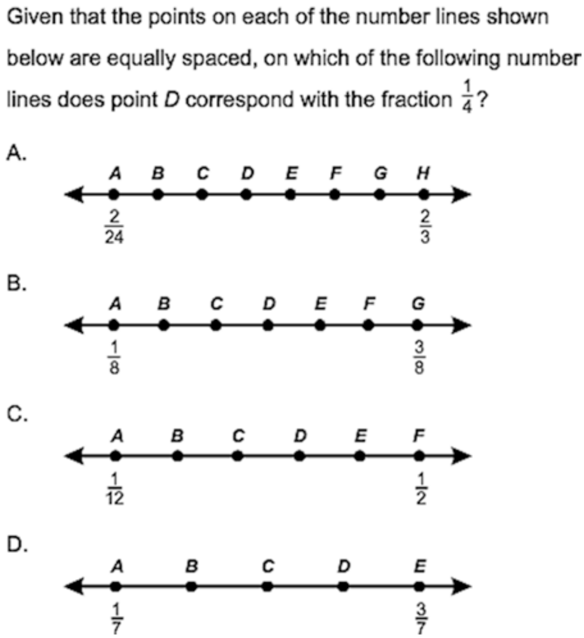


Fig. 5.2 Sample MKT-VR item. (MTEL 2011)

To assemble this MKT-VR assessment, we reviewed existing test banks and assessments including the Educational Testing Services (2013), Learning Mathematics for Teaching assessment (2009), NAEP (2009), and Pearson (2013a, b) to identify items that responded to the types of knowledge and skills that teachers should carry and display to demonstrate CK-VR and PCK-VR when presented with multiple-choice items. We acknowledge that the identified knowledge and skills may be a subset of all the capacities and that may be associated with MKT-VR, but given the scope of this study, we focused on those that may be most easily measured with multiple-choice items. Specifically, the knowledge and skills related to CK-VR are items that ask participants to (a) decode quantities and relationships from a VR, (b) translate a problem into a VR, and (c) find a solution from a VR. An example of a CK-VR item is presented in Fig. 5.2. The knowledge and skills related to PCK-VR are items that ask participants to identify approaches using VR to support students' specific mathematical understanding. The items related to PCK-VR include asking participants to identify approaches using VRs to support students' specific mathematical understanding, including selecting appropriate VRs for teaching specific concepts, and posing appropriate questions of VRs to promote specific understandings.

All items on the MKT-VR assessment are in the content areas of rational number, ratio, and proportion. These topics are critical content areas in middle-grades

ment will be used to test first hypothesis. If pilot testing and expert review indicate that we may have valid and reliable subscales measuring CK-VR and PCK-VR, we may explore treatment vs. control group differences in scores on these subscales as well.

mathematics, as emphasized by the CCSSM, and research suggests that proportionality is a unifying theme across the middle grades (Lanius and Williams 2003). The areas of rational number, ratio, and proportion are also the focus of the VAM PD project. Within these content areas, we selected items that focused on proportional relationships, number lines, and problem solving with fractions and ratios. The assessment will be given to treatment and control teachers, and in analysis, we will compare scores of treatment teachers to scores control teachers, controlling for pretest scores.

MKT-VR Open Response Exercise As we assembled the MKT-VR multiple-choice assessment and reviewed the qualities and utilities of VRs, we determined that our study would benefit from an additional assessment to measure other aspects of VR knowledge, particularly more performance-based skills. This measure of MKT-VR is a measure that is closely related to VAM PD and is only given to treatment teachers, to look for pre-post change (and will not be used to compare treatment and control teachers).

The Open Response Exercise will measure teachers' abilities to identify and describe quantities and relationships in a VR and to compare two solutions involving VRs, by describing the strengths and limitations of each solution and/or the reasoning behind using a specific VR for a task. In the assessment, participants need to (1) read a ratio or proportional reasoning task, (2) review two completed VRs to the ratio or proportional reasoning task, (3) identify and describe the quantities and relationships shown in the different approaches, and (4) provide advantages for the use of each type VR for teaching students about key ratio and proportional reasoning concepts. An excerpt from the Open Response Exercise is presented in the Appendix.

While we considered designing an assessment that would ask participants to create their own VRs to solve a mathematics task, we determined that we did not have the capacity to score what could be a wide range of VRs. Our focus on exploring participants' abilities to identify and describe the mathematical ideas represented in a specific set of VRs and their affordances prompts participants to engage in analysis of VRs, which may support assessment of higher levels of thinking about VRs. This assessment continues to focus on many of the same qualities of VRs as the multiple-choice assessment and emphasizes the utility of VRs while also focusing attention on comparison and evaluation.

The prompts on the Open Response Exercise focus on participants' abilities to identify and discern key quantities and relationships in the VR and to analyze provided VRs for use with a specific mathematics task. In the Open Response Exercise, we seek to understand if and how participants detail and discuss the strengths and limitations of different VR-based approaches for solving the specific problem, including attending to the quantities and relationships shown in the different approaches; alignment and relations among task, VR, and mathematics; and explanation of reasoning about the details of VR use for the mathematics task.

Analysis of teacher responses will be scored using two rubrics. First, teachers' responses on identifying and describing mathematical relationships in a VR used to solve a task will present their ability to describe how to use a VR to solve a ratio and proportional reasoning task (see Appendix). To demonstrate this ability, the participant must use (or describe how to use) quantities and relationships of the VR to solve the task and must link quantities and relationships in the VR to at least one key ratio and proportional reasoning idea. Scoring of responses will differentiate among evidence and no evidence of each component of the ability. Then, teachers' responses on describing if and how to use on VR for teaching students about specific ratio and proportional reasoning concepts will present participants' ability to describe the advantages of using one VR over another for teaching unit rate problems. To demonstrate this ability, the participant must (a) accurately identify *specific* quantities, relationships, or key ratio or proportional reasoning ideas that are more or less visible in one VR than the other and (b) justify by providing one or more reasons for a specific advantage of the tape diagram/double number line for *helping students solve or understand how to solve* unit rate problems with this type of VR. Scoring of responses will differentiate among evidence and no evidence of each component of the ability. We have used the rubrics with pilot data and refined them, coordinating with the VAM project team to make sure that we are measuring the key components and abilities of MKT-VR.

5.7 Discussion and Conclusion

As we seek to measure MKT-VR, we are aware of the multiple purposes and roles of VRs and the connections to mathematical communication, and we are interested in how increased teacher knowledge of VRs, specifically as related to alignment and strategic use of VRs, relates to instructional practices and equitable opportunities to learn: accepting that VRs can support student engagement in mathematics content, tasks, and practices, teacher knowledge likely plays a key role in student *opportunities* to learn about and use VRs. Thus not only do teachers need a robust understanding of the purposes of visually representing mathematics, the roles of VRs in instruction (Stylianou 2010), and relations between accurate and inaccurate VRs to students' problem solving (e.g., Boonen et al. 2014), they need to be able to *coordinate* these different ideas and wrestle with the complexity of their VRs and students' VRs. For example, being strategic with VR use shifts according to contexts and content, and a VR can be a tool in communication even when it may include imprecise number or line placements. To address this complexity of MKT-VR, we are developing multiple measures for understanding teacher MKT-VR, and it will be important to coordinate teachers' responses across measures to better understand the teacher knowledge for mathematics as related to VRs. This paper has laid out a theoretical framework that we anticipate using to guide and benchmark future research.

In addition, research needs to continue to seek to understand how teacher knowledge relates to instruction. We posit that teacher knowledge about VRs and related instructional practice that engages students in the doing of VRs themselves are critical to supporting all students' engagement in mathematics. VRs can be a key engagement and communication tool at the middle grades and can support rich understandings of content; researchers and the CCSSM recommend the use of visual representations in elementary and middle grades, especially number lines and tape diagrams. In addition, VRs are a tool used in mathematics classes for students who are English Learners around the work. VRs were promoted in the CCSSM as writers were influenced not only by the connections to content but by the effective use of VRs in other countries' schools, particularly Japan, China, and Singapore. Since most students in Singapore public schools are native Chinese speakers, who are taught and tested in English, Singapore students' experience and success in learning as measured on international assessments influenced our own work to support mathematics teachers of English Learners, in particular, our work to build teachers' own VR understanding and skills.

To support students who are English Learners, and specifically through opportunities to learn mathematics with VRs, we first need to understand how to measure and supports teachers' representational fluency or their MKT-VR. In this project, we are focused on ratio and proportional reasoning, and related visual representations, seeking to support teacher knowledge in this area (MKT-VR) and their knowledge of instructional practices with VRs. This project is laying the groundwork around future work to investigate whether MKT-VR has other features that should be defined related to different mathematical content, and other ways to support teacher knowledge. Future research should continue to look into how to support and measure teacher knowledge across content areas and grade levels, particularly with an eye to equity and access and promoting mathematical communication and reasoning beyond algorithmic and procedural understandings.

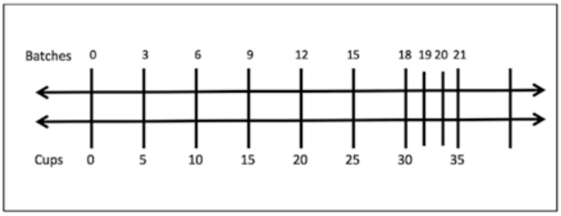
Appendix

Excerpt from MKT-VR Open Response Assessment

Please read the following Cookie Task.

> Andrew has a cookie recipe that uses 5 cups of sugar to make 3 batches of cookies. Draw a visual representation that can help answer the question: How many cups of sugar does Andrew need to make 19 batches of cookies?

1. Ms. Martinez drew the following diagram in response to the task.



   a. List the steps you could take to solve the Cookie Task with this diagram.

   b. What are the mathematical relationships that you see in Ms. Martinez's diagram that allow you to solve the Cookie Task?

2. Mr. Diaz would like to build his sixth grade students' understanding of the mathematics in this standard, "Solve unit rate problems," using a visual representation such as a double number line or tape diagram. (Ms. Martinez's diagram is one example of a double number line. Ms. Bruno's diagram is one example of a tape diagram.) What are the possible advantages of using a double number line compared to using a tape diagram to build students' understanding of the mathematics in the above standard? Please explain as fully as you can.

References

Ball, D. L., Hill, H. H., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator, 29*, 14–46.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special. *Journal of Teacher Education, 59*(5), 389–407.

Boonen, A. J., van Wesel, F., Jolles, J., & van der Schoot, M. (2014). The role of visual representation type, spatial ability, and reading comprehension in word problem solving: An item-level analysis in elementary school children. *International Journal of Educational Research, 68*, 15–26.

Campbell, P. F., Nishio, M., Smith, T. M., Clark, L. M., Conant, D., Rust, A., Neumayer DePiper, J., Jones, T., & Griffin, M. J. (2014). The relationship between teachers' mathematical content and pedagogical knowledge, teachers' perceptions, and student achievement. *Journal for Research in Mathematics Education, 45*(4), 419–459.

Chval, K., & Khisty, L. (2001, April). *Writing in mathematics with Latino students*. Presentation at the annual meeting of the American Educational Research Association, Seattle.

Common Core State Standards Initiative (CCSSI). (2010). *Common core state standards for Mathematics*. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf.

DeVellis, R. F. (2012). *Scale development: Theory and applications* (3rd ed.). Thousand Oaks: SAGE Publications.

DePiper, J. N., Nikula, J., & Driscoll, M. (2014). *Professional development integrating math and language supports for students who are English Learners*. Paper presented at the National Council of Teachers of Mathematics Research Conference, New Orleans, LA.

Educational Testing Services. (2013). Praxis Series.

Gersten, R., Beckmann, S., Clarke, B., Foegen, A., Marsh, L., Star, J. R., & Witzel, B. (2009). *Assisting students struggling with mathematics: Response to intervention (RTI) for elementary and middle schools (NCEE 2009–4060)*. Washington, DC: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S.

Hill, H. C., Blunk, M., Charalambous, C. Y., Lewis, J., Phelps, G. C., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction, 26*, 430–511.

Khisty, L. L., & Chval, K. (2002). Pedagogic discourse and equity in mathematics: When teachers' talk matters. *Mathematics Education Research Journal, 14*(3), 154–168.

Lamon, S. J. (2007). Rational numbers and proportional reasoning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629–667). Charlotte: Information Age Publishing.

Lanius, C., & Williams, S. (2003). Proportionality: A unifying theme for the middle grades. *Mathematics Teaching in the Middle School, 8*, 392–396.

Learning Mathematics for Teaching (LMT). (2009). *Mathematical knowledge for teaching (MKT) measures*. Ann Arbor: University of Michigan.

Lesh, R., Post, T., & Behr, M. (1988). Proportional reasoning. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 93–118). Reston: Lawrence Erlbaum & National Council of Teachers of Mathematics.

Lobato, J., & Ellis, A. B. (2010). Essential understandings: Ratios, proportions, and proportional reasoning. In R. M. Zbiek (Ed.), *Essential understandings*. Reston: National Council of Teachers of Mathematics (NCTM).

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum.

Massachusetts Tests for Educator Licensure (MTEL). (2011). *Middle school mathematics (47) practice test*. Amherst: Pearson Education.

Moschkovich, J. (2002). A situated and sociocultural perspective on bilingual mathematics learners. *Mathematical Thinking & Learning, 4*, 189–212.

National Governors Association Center for Best Practices, & Council of Chief State School Officers (NGA & CCSSO). (2010). Common Core State Standards: Mathematics. Retrieved from http://www.corestandards.org/assets/CCSSI_Math Standards.pdf

Ng, S. F., & Lee, K. (2009). The model method: Singapore children's tool for representing and solving algebraic word problems. *Journal for Research in Mathematics Education, 40*(3), 282–313.

Orrill, C. H., & Brown, R. E. (2012). Making sense of double number lines in professional development: Exploring teachers' understandings of proportional relationships. *Journal of Mathematics Teacher Education, 15*, 381–403.

Pearson Education, Inc. (2013a). *Mathematics teacher educator licensure exam, Mathematics Middle School* (47). Amherst.

Pearson Education, Inc. (2013b). *Mathematics teacher educator licensure exam, Mathematics Elementary* (53). Amherst.

Riley, K. R. (2010). Teachers' understanding of proportional reasoning. In P. Brosnan, D. B. Erchick, & L. Flevares (Eds.), *Proceedings of the 32nd annual meeting of the north American chapter of the International Group for the Psychology of mathematics education* (Vol. 6, pp. 1055–1061). Columbus: The Ohio State University.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*, 4–14.

Siegler, R., Carpenter, T., Fennell, F., Geary, D., Lewis, J., Okamoto, Y., Thompson, L., & Wray, J. (2010). *Developing effective fractions instruction for kindergarten through 8th grade: A practice guide (NCEE #2010-4039)*. Washington, DC: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education Retrieved from http://ies.ed.gov/ncee/wwc/pdf/practice_guides/fractions_pg_093010.pdf.

Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology, 62*(62), 273–296.

Singh, P. (2000). Understanding the concepts of proportion and ratio constructed by two grade six students. *Educational Studies in Mathematics, 43*(3), 271–292.

Smith, J., & Thompson, P. W. (2007). Quantitative reasoning and the development of algebraic reasoning. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 95–132). New York: Erlbaum.

Stylianou, D. A. (2002). On the interaction of visualization and analysis: The negotiation of a visual representation in expert problem solving. *The Journal of Mathematical Behavior, 21*(3), 303–317.

Stylianou, D. A. (2010). Teachers' conceptions of representation in middle school mathematics. *Journal of Math Teacher Education, 13*, 325–343.

Stylianou, D. A. (2011). An examination of middle school students' representation practices in mathematical problem solving through the lens of expert work: towards an organizing scheme. *Educational Studies in Mathematics, 76*, 265–280.

Stylianou, D. A., & Silver, E. A. (2004). The role of visual representations in advanced mathematical problem solving: An examination of expert-novice similarities and differences. *Mathematical Thinking and Learning, 6*(4), 353–387.

U.S. Department of Education, Institute of Education Sciences, National Center for Education Statistics, National Assessment of Educational Progress (NAEP). (2009) Mathematics Assessment. Retrieved from http://nces.ed.gov/NationsReportCard/nqt/DownloadOutput/DownloadOutput/11860.

Woodward, J., Beckmann, S., Driscoll, M., Franke, M., Herzig, P., Jitendra, A., Koedinger, K. R., & Ogbuehi, P. (2012). *Improving mathematical problem solving in grades 4 through 8: A practice guide (NCEE 2012–4055)*. Washington, DC: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education Retrieved from http://ies.ed.gov/ncee/wwc/publications_reviews.aspx#pubsearch/.

Zbiek, R. M., Heid, M. K., & Blume, G. (2007). Research on technology in mathematics education: The perspective of constructs. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 1169–1208). National Council of Teachers of Mathematics: Reston, VA.